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Sébastien Boyaval, Sofiane Martel, Julien Reygner

FINITE-VOLUME APPROXIMATION OF THE INVARIANT MEASURE OF A VISCOUS STOCHASTIC SCALAR CONSERVATION LAW

SÉBASTIEN BOYAVAL, SOFIANE MARTEL, AND JULIEN REYGNER

Abstract. We study the numerical approximation of the invariant measure of a viscous scalar conservation law, one-dimensional and periodic in the space variable, and stochastically forced with a white-in-time but spatially correlated noise. The flux function is assumed to be locally Lipschitz continuous and to have at most polynomial growth. The numerical scheme we employ discretises the SPDE according to a finite-volume method in space, and a split-step backward Euler method in time. As a first result, we prove the well-posedness as well as the existence and uniqueness of an invariant measure for both the semi-discrete and the split-step scheme. Our main result is then the convergence of the invariant measures of the discrete approximations, as the space and time steps go to zero, towards the invariant measure of the SPDE, with respect to the second-order Wasserstein distance. We investigate rates of convergence theoretically, in the case where the flux function is globally Lipschitz continuous with a small Lipschitz constant, and numerically for the Burgers equation.

1. Introduction

1.1. **Viscous scalar conservation law with random forcing.** We consider the following viscous scalar conservation law with stochastic forcing on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\mathrm{d}u = -\partial_x A(u)\mathrm{d}t + \nu\partial_{xx}u\mathrm{d}t + \sum_{k\geq 1} g^k \mathrm{d}W^k(t), \qquad x \in \mathbb{T}, \quad t \geq 0, \tag{1}$$

where $(W^k)_{k\geq 1}$ is a family of independent real Brownian motions and $(g^k)_{k\geq 1}$ is a family of smooth functions on $\mathbb{T}$. The viscosity coefficient ν is assumed to be positive. Under regularity and polynomial growth assumptions on the *flux* function A, Equation (1) is well-posed in a strong sense, and there exists a unique invariant measure for its solution, see Proposition 1.2 below, which is proved in the companion paper [28].

In this work, we construct a numerical scheme, based on the finite-volume method, that allows to approximate this invariant measure. We place ourselves in the setting of [28] and first recall our main notations, assumptions and results.

1.1.1. *Notations.* For any $p \in [1, +\infty]$, we denote by $L^p_0(\mathbb{T})$ the set of functions $v \in L^p(\mathbb{T})$ such that

$$\int_{\mathbb{T}} v(x)\mathrm{d}x = 0,$$

and we write $\|v\|_{L^p_0(\mathbb{T})}$ for the L^p norm induced on $L^p_0(\mathbb{T})$. For any integer $m \geq 0$, we denote by $H^m_0(\mathbb{T})$ the intersection of $L^2_0(\mathbb{T})$ with the Sobolev space $H^m(\mathbb{T})$. Combining the Jensen inequality

$$\forall 1 \leq p \leq q \leq +\infty, \qquad \|v\|_{L^p_0(\mathbb{T})} \leq \|v\|_{L^q_0(\mathbb{T})}, \tag{2}$$

with the gradient estimate

$$\|v\|_{L^\infty_0(\mathbb{T})} \leq \|\partial_x v\|_{L^1_0(\mathbb{T})}, \tag{3}$$

we observe that $\|v\|_{H^m_0(\mathbb{T})} := \|\partial^m_x v\|_{L^2_0(\mathbb{T})}$ defines a norm on $H^m_0(\mathbb{T})$, which is associated with the scalar product $\langle v, w\rangle_{H^m_0(\mathbb{T})}$ and makes $H^m_0(\mathbb{T})$ a separable Hilbert space.

We denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{N}^*$ the set of positive integers.

1.1.2. *Assumptions on the flux and the noise.* We shall assume that the flux function A and the family of functions $(g^k)_{k\geq 1}$ satisfy the following condition.

Assumption 1.1 (On A and $(g^k)_{k\geq 1}$)**.** The function $A : \mathbb{R} \to \mathbb{R}$ is of class C^2, its first derivative has at most polynomial growth:

$$\exists \mathsf{C}_A > 0, \quad \exists \mathsf{p}_A \in \mathbb{N}^*, \quad \forall v \in \mathbb{R}, \qquad |A'(v)| \leq \mathsf{C}_A\left(1 + |v|^{\mathsf{p}_A}\right), \tag{4}$$

and its second derivative A'' is locally Lipschitz continuous on $\mathbb{R}$. Furthermore, for all $k \geq 1$, $g^k \in H^2_0(\mathbb{T})$ and

$$\mathsf{D} := \sum_{k\geq 1} \left\|g^k\right\|^2_{H^2_0(\mathbb{T})} < +\infty. \tag{5}$$

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The assumptions (4) and (5) will be needed in the arguments contained in this paper while the local Lipschitz continuity of A'' is only necessary for Proposition 1.2, which is proved in [28].

The family of Brownian motions $(W^k)_{k\geq 1}$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$ in the sense of [13, Section 3.3]. Under Assumption 1.1, the series $\sum_k g^k W^k$ has to be understood as an $H_0^2(\mathbb{T})$-valued Wiener process W^Q with trace class covariance operator $Q : H_0^2(\mathbb{T}) \to H_0^2(\mathbb{T})$ given by $Qv = \sum_{k\geq 1} g^k \langle v, g^k\rangle_{H_0^2(\mathbb{T})}$, see [28] for details. In the sequel, we shall call W^Q a *Q-Wiener process.*

1.1.3. *Main results from* [28]. Given a normed vector space E, $\mathcal{B}(E)$ denotes the Borel σ-field on E, $\mathcal{P}(E)$ denotes the set of probability measures over $(E, \mathcal{B}(E))$, and for $p \in [1, +\infty)$, $\mathcal{P}_p(E)$ denotes the subset of $\mathcal{P}(E)$ of probability measures with finite p-th order moment. The well-posedness of (1), as well as the existence and uniqueness of an invariant measure for its solution, is proved in [28, Theorem 1, Theorem 2].

Proposition 1.2 (Well-posedness and invariant measure for (1))**.** *Let $u_0 \in H_0^2(\mathbb{T})$. Under Assumption 1.1, there exists a unique strong solution $(u(t))_{t\geq 0}$ to Equation (1) with initial condition u_0. That is, an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(u(t))_{t\geq 0}$ with values in $H_0^2(\mathbb{T})$ such that, almost surely:*

(1) *the mapping $t \mapsto u(t)$ is continuous from $[0, +\infty)$ to $H_0^2(\mathbb{T})$;*
(2) *for all $t \geq 0$, the following equality holds:*

$$u(t) = u_0 + \int_0^t \left(-\partial_x A\left(u(s)\right) + \nu \partial_{xx} u(s)\right) \mathrm{d}s + W^Q(t). \tag{6}$$

Furthermore, the process $(u(t))_{t\geq 0}$ admits a unique invariant measure $\mu \in \mathcal{P}(H_0^2(\mathbb{T}))$, and if v is a random variable with distribution μ, then $\mathbb{E}[\|v\|^2_{H_0^2(\mathbb{T})}] < +\infty$ and for all $p \in [1, +\infty)$, $\mathbb{E}[\|v\|^p_{L_0^p(\mathbb{T})}] < +\infty$.

Let us precise that for any $t \geq 0$, the notation $u(t)$ shall always refer to an element of the space $H_0^2(\mathbb{T})$. The scalar values taken by this function are denoted by $u(t, x)$, for $x \in \mathbb{T}$.

1.2. **Space discretisation.** To discretise (1) with respect to the space variable, we first fix $N \geq 1$, denote by $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ the discrete torus, and define the regular mesh $\mathcal{T}_N$ on $\mathbb{T}$ by

$$\mathcal{T}_N := \left\{(x_{i-1}, x_i], i \in \mathbb{T}_N\right\}, \qquad x_i := \frac{i}{N},$$

where we identify $\mathbb{T}$ with $(0, 1]$ and $\mathbb{T}_N$ with $\{1, \dots, N\}$. Next, we introduce the finite dimensional space

$$\mathbb{R}_0^N := \{\mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}^N : v_1 + \cdots + v_N = 0\},$$

on which we define, for any $p \in [1, +\infty]$, the normalised ℓ^p norm

$$\|\mathbf{v}\|_{\ell_0^p(\mathbb{T}_N)} := \left(\frac{1}{N}\sum_{i\in\mathbb{T}_N} |v_i|^p\right)^{1/p} \quad \text{if } p < +\infty, \qquad \|\mathbf{v}\|_{\ell_0^\infty(\mathbb{T}_N)} := \max_{i\in\mathbb{T}_N} |v_i|.$$

The *projection operator* $\Pi_N : L_0^1(\mathbb{T}) \to \mathbb{R}_0^N$ is defined by

$$\forall i \in \mathbb{T}_N, \qquad (\Pi_N v)_i = N \int_{x_{i-1}}^{x_i} v(x)\mathrm{d}x.$$

Notice that by Jensen's inequality, it satisfies the inequality

$$\|\Pi_N v\|_{\ell_0^p(\mathbb{T}_N)} \leq \|v\|_{L_0^p(\mathbb{T})}, \tag{7}$$

for any $p \in [1, +\infty]$.

Applying this operator to both sides of (1), we get, for any $i \in \mathbb{T}_N$,

$$\mathrm{d}\left(\Pi_N u(t)\right)_i = -N\left(A\left(u\left(t, x_i\right)\right) - A\left(u\left(t, x_{i-1}\right)\right)\right)\mathrm{d}t + \nu N\left(\partial_x u\left(t, x_i\right) - \partial_x u\left(t, x_{i-1}\right)\right)\mathrm{d}t + (\Pi_N W^Q(t))_i.$$

Let us denote by $\mathbf{U}^N(t) = (U_i^N(t))_{i\in\mathbb{T}_N}$ a vector whose purpose is to approximate the vector $\Pi_N u(t)$. The basic idea of finite-volume schemes consists in approximating the flux function $A(u(t, x_i))$ at the interface between two adjacent cells by a *numerical flux* $\overline{A}(U_i^N(t), U_{i+1}^N(t))$, where the function $\overline{A} : \mathbb{R}^2 \to \mathbb{R}$ satisfies certain properties which are stated in Assumption 1.3 below. Given such a function, for any $\mathbf{v} \in \mathbb{R}_0^N$ we denote by $\overline{\mathbf{A}}^N(\mathbf{v})$ the vector with coordinates $\overline{A}(v_i, v_{i+1})$, $i \in \mathbb{T}_N$.

We then introduce the first-order forward and backward discrete derivative operators $\mathbf{D}_N^{(1,+)}$ and $\mathbf{D}_N^{(1,-)}$, defined by

$$\forall i \in \mathbb{T}_N, \qquad (\mathbf{D}_N^{(1,+)}\mathbf{v})_i := N(v_{i+1} - v_i), \quad (\mathbf{D}_N^{(1,-)}\mathbf{v})_i := N(v_i - v_{i-1}),$$

and the second-order centered discrete derivative operator $\mathbf{D}_N^{(2)}$, defined by

$$\mathbf{D}_N^{(2)}\mathbf{v} = \mathbf{D}_N^{(1,-)}\mathbf{D}_N^{(1,+)}\mathbf{v} = \mathbf{D}_N^{(1,+)}\mathbf{D}_N^{(1,-)}\mathbf{v}.$$

In the sequel, we shall sometimes call the quantities $\|\mathbf{D}^{(1,+)}\mathbf{v}\|_{\ell_0^2(\mathbb{T})}$ and $\|\mathbf{D}^{(2)}\mathbf{v}\|_{\ell_0^2(\mathbb{T})}$ the $h_0^1(\mathbb{T}_N)$ and $h_0^2(\mathbb{T}_N)$ norms of $\mathbf{v}$.

For all $k \geq 1$, we finally define the vector $\mathbf{g}^k := \Pi_N g^k \in \mathbb{R}_0^N$ and denote by $\mathbf{W}^{Q,N}$ the process $\Pi_N W^Q = \sum_k \mathbf{g}^k W^k$. This is a Wiener process in $\mathbb{R}_0^N$ with covariance

$$\mathbb{E}\left[W_i^{Q,N}(t)W_j^{Q,N}(t)\right] = t\sum_{k\geq 1} g_i^k g_j^k, \tag{8}$$

which is easily seen to be finite under Assumption 1.1 (see (24) below).

These notations allow us to write a semi-discrete finite-volume approximation of (1) as the stochastic differential equation (SDE)

$$\mathrm{d}\mathbf{U}^N(t) = -\mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(t))\mathrm{d}t + \nu\mathbf{D}_N^{(2)}\mathbf{U}^N(t)\mathrm{d}t + \mathrm{d}\mathbf{W}^{Q,N}(t). \tag{9}$$

We shall sometimes use the notation

$$\mathbf{b}(\mathbf{v}) := -\mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{v}) + \nu\mathbf{D}_N^{(2)}\mathbf{v}$$

for the drift of this SDE. Since both this vector field and the noise $\mathbf{W}^{Q,N}$ take their values in $\mathbb{R}_0^N$, we deduce that (9) is conservative in the sense that if $\mathbf{U}^N(0) \in \mathbb{R}_0^N$, then for all $t \geq 0$, $\mathbf{U}^N(t) \in \mathbb{R}_0^N$.

We may now state our assumptions on the numerical flux.

Assumption 1.3 (On $\overline{A}$)**.** The function $\overline{A}$ belongs to $C^1(\mathbb{R}^2, \mathbb{R})$, its first derivatives $\partial_1\bar{A}$ and $\partial_2\bar{A}$ are locally Lipschitz continuous on $\mathbb{R}^2$, and it satisfies the following properties.

(i) Consistency:

$$\forall v \in \mathbb{R}, \qquad \overline{A}(v,v) = A(v). \tag{10}$$

(ii) Monotonicity:

$$\forall v,w \in \mathbb{R}, \qquad \partial_1\bar{A}(v,w) \geq 0, \quad \partial_2\bar{A}(v,w) \leq 0. \tag{11}$$

(iii) Polynomial growth:

$$\exists \mathsf{C}_{\bar{A}} > 0, \quad \exists \mathsf{p}_{\bar{A}} \in \mathbb{N}^*, \quad \forall v,w \in \mathbb{R}, \quad |\partial_1\overline{A}(v,w)| \leq \mathsf{C}_{\bar{A}}(1+|v|^{\mathsf{p}_{\bar{A}}}), \quad |\partial_2\overline{A}(v,w)| \leq \mathsf{C}_{\bar{A}}(1+|w|^{\mathsf{p}_{\bar{A}}}). \tag{12}$$

Note in particular that the numerical flux function, and therefore the drift of the SDE (9), is not globally Lipschitz continuous. Nevertheless, we prove in Proposition 2.4 that (9) is well-posed under Assumption 1.3.

Remark 1.4 (Engquist–Osher numerical flux)**.** A notable class of numerical fluxes satisfying the monotonicity and polynomial growth conditions (under Assumption 1.1) are the flux-splitting schemes [21, Example 5.2], among which a commonly employed example is the *Engquist–Osher flux* [20] defined (for $A(0)=0$) by

$$\forall v,w \in \mathbb{R}, \qquad \bar{A}_{\mathrm{EO}}(v,w) := \int_0^v [A'(v')]_+ \mathrm{d}v' - \int_0^w [A'(w')]_- \mathrm{d}w'.$$

1.3. **Space and time discretisation.** The second stage in constructing a numerical scheme for (1) is the time discretisation of the SDE (9). Considering a time step $\Delta t > 0$ and a positive integer n, we introduce the notation

$$\Delta\mathbf{W}_n^{Q,N} := \mathbf{W}^{Q,N}(n\Delta t) - \mathbf{W}^{Q,N}((n-1)\Delta t). \tag{13}$$

As already noticed in [29], explicit numerical schemes for SDEs with non-globally Lipschitz continuous coefficients do not preserve in general the long time stability, whereas implicit schemes are more robust. Therefore, since our main focus in this paper is to approximate invariant measures, we follow [29] and propose the following *split-step stochastic backward Euler method*

$$\begin{cases} \mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t} = \mathbf{U}_n^{N,\Delta t} + \Delta t\mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right), \\ \mathbf{U}_{n+1}^{N,\Delta t} = \mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t} + \Delta\mathbf{W}_{n+1}^{Q,N}. \end{cases} \tag{14}$$

The well-posedness of the scheme, *i.e.* the existence and uniqueness of the value $\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}$ in the first line of (14), is ensured by Proposition 2.13.

1.4. **Main results.** Our first focus is on the long time behaviour of the processes $(\mathbf{U}^N(t))_{t\geq 0}$ and $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$. In this perspective, we state our first result.

Theorem 1.5 (Existence and uniqueness of invariant measures for both schemes)**.** *Under Assumptions 1.1 and 1.3, the following two statements hold:*

(i) *for any $N \geq 1$, the process $(\mathbf{U}^N(t))_{t\geq 0}$ solution of the SDE (9) admits a unique invariant measure $\nu_N \in \mathcal{P}(\mathbb{R}_0^N)$;*

(ii) *for any $N \geq 1$ and $\Delta t > 0$, the sequence $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ defined by (14) admits a unique invariant measure $\nu_{N,\Delta t} \in \mathcal{P}(\mathbb{R}_0^N)$.*

Moreover, for any $N \geq 1$ and $\Delta t > 0$, the measures ν_N and $\nu_{N,\Delta t}$ belong to $\mathcal{P}_2(\mathbb{R}_0^N)$.

The proofs for these two statements are given separately in Section 2. The structure of the proof is the same as for [28, Theorem 2] where we derived the existence and uniqueness of an invariant measure for the solution of (1) from two important properties: respectively the dissipativity of the $L^2_0(\mathbb{T})$ norm of the solution, and an $L^1_0(\mathbb{T})$ contraction property. In Lemma 2.1 below, we show that both of these properties are preserved at the discrete level. Therefore, we then prove the existence of an invariant measure with a tightness argument (uniform energy estimates and the Krylov–Bogoliubov theorem) and the uniqueness with a coupling argument. While the proof of existence is rather standard, our analysis of uniqueness depends on arguments which are more specific to the processes $(\mathbf{U}^N(t))_{t\geq 0}$ and $(\mathbf{U}^{N,\Delta t}_n)_{n\in\mathbb{N}}$. We insist on the fact that both proofs of existence and uniqueness crucially rely on the positivity of the viscosity coefficient ν.

We now address the $\Delta t \to 0$ limit of $\nu_{N,\Delta t}$ and the $N \to +\infty$ limit of ν_N. Since most of our results follow from $\ell^2_0(\mathbb{T}_N)$ or $L^2_0(\mathbb{T})$ estimates, it is natural in our setting to work with the following distance on $\mathcal{P}_2(\ell^2_0(\mathbb{T}_N))$ or $\mathcal{P}_2(L^2_0(\mathbb{T}))$.

Definition 1.6 (Wasserstein distance)**.** Let $(E, \|\cdot\|_E)$ be a normed vector space and let $\alpha, \beta \in \mathcal{P}_2(E)$. The *quadratic Wasserstein distance* between α and β is defined by

$$W_2(\alpha,\beta) := \inf_{\pi\in\Pi(\alpha,\beta)} \left(\int_{E\times E} \|u-v\|^2_E \,\mathrm{d}\pi(u,v)\right)^{1/2},$$

where $\Pi(\alpha,\beta)$ is the set of probability measures on $E\times E$ with marginals α and β:

$$\Pi(\alpha,\beta) := \{\pi \in \mathcal{P}_2(E\times E) : \forall B \in \mathcal{B}(E), \pi(B\times E) = \alpha(B) \text{ and } \pi(E\times B) = \beta(B)\}.$$

The reader is referred to [31, Chapter 6] for further details on the Wasserstein distance, and in particular for the proof that it actually defines a distance on $\mathcal{P}_2(E)$. From now on, the spaces $\mathcal{P}_2(\ell^2_0(\mathbb{T}_N))$ and $\mathcal{P}_2(L^2_0(\mathbb{T}))$ are systematically endowed with the topology induced by the corresponding distance W_2.

As a first step to approximate numerically the measure μ, we embed the measures ν_N and $\nu_{N,\Delta t}$ into $\mathcal{P}(L^2_0(\mathbb{T}))$. Let us denote by $\Psi_N : \mathbb{R}^N_0 \to L^\infty_0(\mathbb{T})$ the piecewise constant reconstruction operator defined by, for all $\mathbf{v}\in\mathbb{R}^N_0$,

$$\forall i \in \mathbb{T}_n, \quad \forall x \in (x_{i-1}, x_i], \qquad \Psi_N\mathbf{v}(x) := v_i.$$

Notice that for any $p \in [1,+\infty]$,

$$\|\Psi_N\mathbf{v}\|_{L^p_0(\mathbb{T})} = \|\mathbf{v}\|_{\ell^p_0(\mathbb{T}_N)}, \tag{15}$$

so that Theorem 1.5 implies that the pushforward measures

$$\mu_N := \nu_N \circ (\Psi_N)^{-1}, \qquad \mu_{N,\Delta t} := \nu_{N,\Delta t} \circ (\Psi_N)^{-1}, \tag{16}$$

belong to $\mathcal{P}_2(L^2_0(\mathbb{T}))$. Sections 3 and 4 are devoted to the proof of our main result.

Theorem 1.7 (Convergence of the invariant measures)**.** *Under Assumptions 1.1 and 1.3, we have*

$$\lim_{N\to\infty} \mu_N = \mu \qquad \text{in } \mathcal{P}_2(L^2_0(\mathbb{T})), \tag{17}$$

and moreover, for any $N \geq 1$,

$$\lim_{\Delta t\to 0} \nu_{N,\Delta t} = \nu_N \qquad \text{in } \mathcal{P}_2(\mathbb{R}^N_0). \tag{18}$$

In short, we have the following approximation result:

$$\lim_{N\to\infty}\lim_{\Delta t\to 0} \mu_{N,\Delta t} = \mu \qquad \text{in } \mathcal{P}_2(L^2_0(\mathbb{T})).$$

Remark 1.8. In Theorem 1.7, μ is seen as a probability measure of $\mathcal{P}(L^2_0(\mathbb{T}))$ giving full weight to $H^2_0(\mathbb{T})$, as opposed to Proposition 1.2 where μ was seen as a probability measure of $\mathcal{P}(H^2_0(\mathbb{T}))$. The fact that both objects coincide follows from [28, Lemma 6].

Let us briefly sketch the lines of our proof of (17). First, the positivity of the viscosity coefficient ν makes (1) parabolic, so that energy estimates in $L^2_0(\mathbb{T})$ and $H^1_0(\mathbb{T})$ are a natural tool to study this equation. In this perspective, we derive uniform (in N) discrete $\ell^p_0(\mathbb{T}_N)$, $h^1_0(\mathbb{T}_N)$ and $h^2_0(\mathbb{T}_N)$ bounds on ν_N. They imply that the sequence $(\mu_N)_{N\geq 1}$ is relatively compact in $\mathcal{P}_2(L^2_0(\mathbb{T}))$. Using the finite-time convergence of the finite-volume scheme $\Psi_N\mathbf{U}^N(t)$ to the solution $u(t)$ of the SPDE (1), we then show that any limit μ^* of a weakly converging subsequence of $(\mu_N)_{N\geq 1}$ is invariant for (1), which allows to identify all these limits and leads to (17). This finite-time convergence result, which is an important step in our argument, is stated in Proposition 3.4. It relies in particular on $H^2_0(\mathbb{T})$ estimates on u, so that the framework of *strong* solutions for (1) is well-suited to our approach.

The proof of (18) follows the same approach, and the main finite-time convergence result is stated in Proposition 4.3.

Remark 1.9 (Ergodicity)**.** As the invariant measure μ of the process $(u(t))_{t\geq 0}$ is unique from Proposition 1.2, it is ergodic. In particular, a consequence of Birkhoff's ergodic theorem (see for instance [13, Theorem 1.2.3]) is that for any $\varphi \in L^1(\mu)$ and for μ-almost every initial condition $u_0 \in H^2_0(\mathbb{T})$, almost surely,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(u(s))\mathrm{d}s = \mathbb{E}\left[\varphi(v)\right], \qquad \text{where } v \sim \mu.$$

By virtue of Theorem 1.5, this property also holds at the discrete level: the sequence $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ satisfies for any $\varphi \in L^1(\nu_{N,\Delta t})$ and for $\nu_{N,\Delta t}$-almost every initial condition $\mathbf{U}_0^{N,\Delta t} \in \mathbb{R}_0^N$, almost surely,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{l=0}^{n-1} \varphi(\mathbf{U}_l^{N,\Delta t}) = \mathbb{E}\left[\varphi(\mathbf{V}^{N,\Delta t})\right], \qquad \text{where } \mathbf{V}^{N,\Delta t} \sim \nu_{N,\Delta t}.$$

Thanks to this property, it is possible to approximate numerically expectations of functionals under the invariant measure by averaging in time the simulated process. We used this method to perform the numerical experiments presented in Section 5.

Complementing the convergence results of Theorem 1.7 with a quantitative rate in N and Δt is a natural question. As far as the convergence of μ_N to μ when $N \to +\infty$ is concerned, we sketch in Subsection 3.4 how our arguments may be adapted to yield a strong $L_0^2(\mathbb{T})$ error estimate of order $1/N$ between $u^N(t)$ and $u(t)$, valid in the long time limit, in the case where the flux function A is globally Lipschitz continuous with a small Lipschitz norm. This implies that $W_2(\mu_N, \mu) \leq C/N$. We show in Section 5 that this result is sharp in the case where $A = 0$. In this section, we also study numerically the rate of convergence of $\nu_{N,\Delta t}$ to ν_N and observe a weak error of order Δt, even when the flux function is not small and not Lipschitz continuous, which is consistent with theoretical results by Kopec on a related class of split-step schemes [25].

1.5. **Review of literature.** Many results are found concerning the numerical approximation *in finite time* of stochastic conservation laws. A particular case of interest is the stochastic Burgers equation which corresponds to the case of the flux function $A(v) = v^2/2$. Finite difference schemes are presented in [1, 24] to approximate its solution. When the viscosity coefficient is equal to zero, the SPDE falls into a different framework. Convergence of finite-volume schemes in this *hyperbolic* case have been established both under the kinetic [17, 18, 16] and the entropic formulations [2, 3].

As regards the numerical approximation of the invariant measure of an SPDE, we may start by mentioning [10] concerning the damped stochastic non-linear Schrödinger equation, where a spectral Galerkin method is used for the space discretisation and a modified implicit Euler scheme for the temporal discretisation. Several works by Bréhier and coauthors are devoted to the numerical approximation of the invariant measures of semi-linear SPDEs in Hilbert spaces perturbed with white noise [5, 6, 7], where spectral Galerkin and semi-implicit Euler methods are used. Those results hold under a global Lipschitz assumption on the nonlinearity. In the more recent works [11, 12, 8], non-Lipschitz nonlinearities are considered, but they still need to satisfy a one-sided Lipschitz condition.

In the present work, our assumptions on the flux function do not imply that the non-linear term is globally Lipschitz continuous in $L_0^2(\mathbb{T})$ nor even one-sided Lipschitz continuous. In particular, the case of the Burgers equation is covered. However, Equation (1) satisfies an $L_0^1(\mathbb{T})$ contraction property [28, Proposition 5] which may be viewed as a one-sided Lipschitz condition in the Banach space $L_0^1(\mathbb{T})$.

1.6. **Outline of the paper and comments on the presentation.** Throughout the article, we always work under Assumptions 1.1 and 1.3. We will not repeat these assumptions in the statements of our results.

Section 2 is dedicated to the proof of the well-posedness and of the existence and uniqueness of an invariant measure for the semi-discrete scheme (9) and the split-step scheme (14). The proof of Theorem 1.7 is then detailed in two separate sections. The convergence in space (17) is proved in Section 3 and then, in Section 4, we prove the convergence with respect to the time step, *i.e.* Equation (18). Numerical experiments investigating the rates of convergence in Theorem 1.7 are presented in Section 5. The proofs of certain results which are not essential to the exposition of our arguments are gathered in Appendix.

In order to emphasise our original contributions, throughout the paper some arguments which are standard to either stochastic calculus or numerical analysis are omitted or merely sketched. A preliminary version of this work, with all proofs detailed, is available as Chapter 3 of [27], and references to this document are given whenever necessary (see also the first arXiv version of this paper). More numerical experiments, in particular regarding the turbulent behaviour of the process in its stationary regime, are also reported in Chapter 4 of [27].

2. Semi-discrete and split-step schemes: well-posedness and invariant measure

Preliminary results are given in Subsection 2.1. In Subsection 2.2, we prove the well-posedness of the semi-discrete scheme $(\mathbf{U}^N(t))_{t\geq 0}$, and after establishing some properties of this process, we prove the existence and uniqueness of an invariant measure ν_N as well as the fact that necessarily, $\nu_N \in \mathcal{P}_2(\mathbb{R}_0^N)$. Similar results for the split-step scheme $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ are obtained in Subsection 2.3, which completes the proof of Theorem 1.5.

2.1. **Preliminary results.** In this subsection, we state a few preliminary results which will be used throughout the paper.

2.1.1. *Algebraic identities.* We define the normalised scalar product on $\mathbb{R}^N$ by

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\ell^2(\mathbb{T}_N)} = \frac{1}{N} \sum_{i \in \mathbb{T}_N} v_i w_i.$$

When both $\mathbf{v}$ and $\mathbf{w}$ belong to $\mathbb{R}^N_0$, we shall rather denote $\langle \mathbf{v}, \mathbf{w} \rangle_{\ell^2_0(\mathbb{T}_N)}$.

The discrete derivative operators $\mathbf{D}_N^{(1,+)}$ and $\mathbf{D}_N^{(1,-)}$ satisfy the summation by parts identity

$$\langle \mathbf{D}_N^{(1,+)} \mathbf{v}, \mathbf{w} \rangle_{\ell^2(\mathbb{T}_N)} = -\langle \mathbf{v}, \mathbf{D}_N^{(1,-)} \mathbf{w} \rangle_{\ell^2(\mathbb{T}_N)}. \tag{19}$$

We shall also use the variant

$$\langle \mathbf{D}_N^{(1,+)} \mathbf{v}, \mathbf{D}_N^{(1,+)} \mathbf{w} \rangle_{\ell^2_0(\mathbb{T}_N)} = \langle \mathbf{D}_N^{(1,-)} \mathbf{v}, \mathbf{D}_N^{(1,-)} \mathbf{w} \rangle_{\ell^2_0(\mathbb{T}_N)} = -\langle \mathbf{D}_N^{(2)} \mathbf{v}, \mathbf{w} \rangle_{\ell^2(\mathbb{T}_N)}. \tag{20}$$

2.1.2. *Discrete inequalities.* The discrete Jensen inequality writes

$$\forall 1 \leq p \leq q \leq +\infty, \qquad \|\mathbf{v}\|_{\ell^p_0(\mathbb{T}_N)} \leq \|\mathbf{v}\|_{\ell^q_0(\mathbb{T}_N)}, \tag{21}$$

and we shall use the following version of the discrete Poincaré inequality:

$$\|\mathbf{v}\|_{\ell^2_0(\mathbb{T}_N)} \leq \|\mathbf{D}_N^{(1,+)} \mathbf{v}\|_{\ell^2_0(\mathbb{T}_N)} = \|\mathbf{D}_N^{(1,-)} \mathbf{v}\|_{\ell^2_0(\mathbb{T}_N)}, \tag{22}$$

which follows from (21) and

$$\|\mathbf{v}\|_{\ell^\infty_0(\mathbb{T}_N)} \leq \|\mathbf{D}_N^{(1,+)} \mathbf{v}\|_{\ell^1_0(\mathbb{T}_N)}, \tag{23}$$

which is the discrete version of the gradient estimate (3).

2.1.3. *Properties of* $\mathbf{b}$. For any $z \in \mathbb{R}$, we write $\mathrm{sign}(z) := \mathbb{1}_{\{z \geq 0\}} - \mathbb{1}_{\{z<0\}}$. By extension, for $\mathbf{v} \in \mathbb{R}^N_0$, $\mathbf{sign}(\mathbf{v})$ denotes the vector of $\{-1,+1\}^N$ defined by $(\mathbf{sign}(\mathbf{v}))_i = \mathrm{sign}(v_i)$. The discretised drift $\mathbf{b}$ preserves two important features of Equation (1) that we will use repeatedly throughout this paper:

Lemma 2.1 (Discrete contraction and dissipativity)**.** *For all* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^N_0$*, the function* $\mathbf{b}$ *satisfies*

(i) $\langle \mathbf{sign}(\mathbf{v}-\mathbf{w}), \mathbf{b}(\mathbf{v}) - \mathbf{b}(\mathbf{w}) \rangle_{\ell^2(\mathbb{T}_N)} \leq 0$ *($\ell^1_0(\mathbb{T}_N)$ contraction);*

(ii) $\langle \mathbf{v}, \mathbf{b}(\mathbf{v}) \rangle_{\ell^2_0(\mathbb{T}_N)} \leq -\nu \|\mathbf{D}_N^{(1,+)} \mathbf{v}\|^2_{\ell^2_0(\mathbb{T}_N)}$ *($\ell^2_0(\mathbb{T}_N)$ dissipativity).*

The proof of Lemma 2.1 relies on the following result, the proof of which is postponed to Appendix A.

Lemma 2.2 (Stability)**.** *For any* $\mathbf{v} \in \mathbb{R}^N_0$ *and any* $q \in 2\mathbb{N}^*$*, we have*

$$\langle \mathbf{v}^{q-1}, \mathbf{D}^{(1,-)} \overline{\mathbf{A}}^N(\mathbf{v}) \rangle_{\ell^2(\mathbb{T}_N)} \geq 0,$$

where the notation $\mathbf{v}^{q-1}$ *refers to the vector with coordinates* $(v_1^{q-1}, \ldots, v_N^{q-1})$*.*

We now detail the proof of Lemma 2.1.

Proof of Lemma 2.1. (i) Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^N_0$. From the definition of $\mathbf{b}$ and (19–20), we write

$$\begin{aligned}
&\langle \mathbf{sign}(\mathbf{v}-\mathbf{w}), \mathbf{b}(\mathbf{v}) - \mathbf{b}(\mathbf{w}) \rangle_{\ell^2(\mathbb{T}_N)} \\
&= -\langle \mathbf{sign}(\mathbf{v}-\mathbf{w}), \mathbf{D}_N^{(1,-)}(\overline{\mathbf{A}}^N(\mathbf{v}) - \overline{\mathbf{A}}^N(\mathbf{w})) \rangle_{\ell^2(\mathbb{T}_N)} + \nu \langle \mathbf{sign}(\mathbf{v}-\mathbf{w}), \mathbf{D}_N^{(2)}(\mathbf{v}-\mathbf{w}) \rangle_{\ell^2(\mathbb{T}_N)} \\
&= \langle \mathbf{D}_N^{(1,+)} \mathbf{sign}(\mathbf{v}-\mathbf{w}), \overline{\mathbf{A}}^N(\mathbf{v}) - \overline{\mathbf{A}}^N(\mathbf{w}) \rangle_{\ell^2(\mathbb{T}_N)} - \nu \langle \mathbf{D}_N^{(1,+)} \mathbf{sign}(\mathbf{v}-\mathbf{w}), \mathbf{D}_N^{(1,+)}(\mathbf{v}-\mathbf{w}) \rangle_{\ell^2_0(\mathbb{T}_N)}.
\end{aligned}$$

Observe that since the function $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$ is non-decreasing,

$$\langle \mathbf{D}_N^{(1,+)} \mathbf{sign}(\mathbf{v}-\mathbf{w}), \mathbf{D}_N^{(1,+)}(\mathbf{v}-\mathbf{w}) \rangle_{\ell^2_0(\mathbb{T}_N)} \geq 0.$$

As for the other term, it follows from the monotonicity property of $\bar{A}$ that for any $i \in \mathbb{T}_N$,

$$(\mathrm{sign}(v_{i+1} - w_{i+1}) - \mathrm{sign}(v_i - w_i)) \left(\overline{A}(v_i, v_{i+1}) - \overline{A}(w_i, w_{i+1}) \right) \leq 0.$$

Indeed, let us address for instance the case where $v_{i+1} \geq w_{i+1}$ and $v_i \leq w_i$. Then, on the one hand, we have $\mathrm{sign}(v_{i+1} - w_{i+1}) - \mathrm{sign}(v_i - w_i) = 2$. On the other hand, we have

$$\begin{aligned}
\bar{A}(v_i, v_{i+1}) - \bar{A}(w_i, w_{i+1}) &= \left(\bar{A}(v_i, v_{i+1}) - \bar{A}(v_i, w_{i+1}) \right) + \left(\bar{A}(v_i, w_{i+1}) - \bar{A}(w_i, w_{i+1}) \right) \\
&= \int_{w_{i+1}}^{v_{i+1}} \partial_2 \bar{A}(v_i, z) \mathrm{d}z - \int_{v_i}^{w_i} \partial_1 \bar{A}(z, w_{i+1}) \mathrm{d}z \leq 0.
\end{aligned}$$

The case where $v_{i+1} \leq w_{i+1}$ and $v_i \geq w_i$ is treated symmetrically.

(ii) Let $\mathbf{v} \in \mathbb{R}^N_0$. We have

$$\langle \mathbf{v}, \mathbf{b}(\mathbf{v}) \rangle_{\ell^2_0(\mathbb{T}_N)} = -\langle \mathbf{v}, \mathbf{D}_N^{(1,-)} \overline{\mathbf{A}}^N(\mathbf{v}) \rangle_{\ell^2_0(\mathbb{T}_N)} + \nu \langle \mathbf{v}, \mathbf{D}_N^{(2)} \mathbf{v} \rangle_{\ell^2_0(\mathbb{T}_N)}.$$

Lemma 2.2 with $q = 2$ shows that the first term of the above decomposition is non-positive, and applying (20) in the second term yields the result. □

Remark 2.3. The $\ell_0^2(\mathbb{T}_N)$ dissipativity property actually holds for the family of E-fluxes [26], a larger family than the class of monotone numerical fluxes. The monotonicity assumption (11) seems however necessary as regards the $\ell_0^1(\mathbb{T}_N)$ contraction property.

2.1.4. *Finiteness of the covariance of* $\mathbf{W}^{Q,N}$. At several places we shall need the estimates

$$\max_{i\in\mathbb{T}_N}\sum_{k\geq 1}(g_i^k)^2 \leq \mathsf{D}, \qquad \sum_{k\geq 1}\|\mathbf{g}^k\|^2_{\ell_0^2(\mathbb{T}_N)} \leq \mathsf{D}, \qquad \sum_{k\geq 1}\|\mathbf{D}_N^{(1,+)}\mathbf{g}^k\|^2_{\ell_0^2(\mathbb{T}_N)} \leq \mathsf{D}, \tag{24}$$

which follow from (2), (3) and (5), and prove for instance the finiteness of the sum in the right-hand side of (8).

2.2. **The semi-discrete scheme.** In this subsection, we first show that the SDE (9) has a unique global solution $(\mathbf{U}^N(t))_{t\geq 0}$. We then give uniform $\ell_0^p(\mathbb{T}_N)$ estimates on this process, which will be used at several places in the sequel of the paper. We finally prove the existence and the uniqueness of an invariant measure ν_N for $(\mathbf{U}^N(t))_{t\geq 0}$.

2.2.1. *Well-posedness of* (9). Since the function $\mathbf{b}$ is locally Lipschitz continuous, it is a standard result that there exists a unique strong solution $(\mathbf{U}^N(t))_{t\in[0,T^*)}$ to Equation (9) defined up to a random explosion time T^*. That this solution is actually global in time usually follows from a Lyapunov-type condition. In our context, the presence of a viscous term in the SPDE (1) allows the use of energy methods based on the dissipation of the squared $L_0^2(\mathbb{T})$ norm, see [28]. At the level of the SDE (9), denoting by $\mathcal{L}_N$ the associated infinitesimal generator, this fact is observed on the simple estimate

$$\mathcal{L}_N\|\mathbf{v}\|^2_{\ell_0^2(\mathbb{T}_N)} = 2\langle \mathbf{v}, \mathbf{b}(\mathbf{v})\rangle_{\ell_0^2(\mathbb{T}_N)} + \sum_{k\geq 1}\|\mathbf{g}^k\|^2_{\ell_0^2(\mathbb{T}_N)} \leq -2\nu\|\mathbf{D}_N^{(1,+)}\mathbf{v}\|^2_{\ell_0^2(\mathbb{T}_N)} + \mathsf{D} \leq -2\nu\|\mathbf{v}\|^2_{\ell_0^2(\mathbb{T}_N)} + \mathsf{D}, \tag{25}$$

which follows from Lemma 2.1, (24) and (22). This shows that the squared $\ell_0^2(\mathbb{T}_N)$ norm is a *Lyapunov function* for $\mathcal{L}_N$, and implies the following statement.

Proposition 2.4 (Well-posedness of (9))**.** *Let $\mathbf{U}_0^N$ be an $\mathbb{R}_0^N$-valued, $\mathcal{F}_0$-measurable random variable. The stochastic differential equation* (9) *admits a unique strong solution $(\mathbf{U}^N(t))_{t\geq 0}$ taking values in $\mathbb{R}_0^N$ and with initial condition $\mathbf{U}_0^N$.*

The proof of Proposition 2.4 is omitted, we refer to [27, Proposition 3.15] for details.

2.2.2. *Moment estimates.* In this paragraph, we prove the following uniform (in N) $\ell_0^p(\mathbb{T}_N)$ estimates on the process $\mathbf{U}^N$. For any $\mathbf{v}\in\mathbb{R}_0^N$, we recall the notation $\mathbf{v}^p = (v_1^p, \dots, v_N^p)$ and take the convention that $\|\mathbf{v}\|^0_{\ell_0^0(\mathbb{T}_N)} = 1$.

Lemma 2.5 (Moment estimates on the semi-discrete scheme)**.** *Let $p \in 2\mathbb{N}^*$ and let $\mathbf{U}_0^N$ be an $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}[\|\mathbf{U}_0^N\|^p_{\ell_0^p(\mathbb{T}_N)}] < +\infty$. The solution $(\mathbf{U}^N(t))_{t\geq 0}$ of* (9) *with initial condition $\mathbf{U}_0^N$ satisfies the following estimates.*

(i) *For all $t \geq 0$,*

$$\begin{aligned}&\mathbb{E}\left[\|\mathbf{U}^N(t)\|^p_{\ell_0^p(\mathbb{T}_N)}\right] + \nu p\mathbb{E}\left[\int_0^t \left\langle \mathbf{D}_N^{(1,+)}\left((\mathbf{U}^N(s))^{p-1}\right), \mathbf{D}_N^{(1,+)}\mathbf{U}^N(s)\right\rangle_{\ell_0^2(\mathbb{T}_N)} \mathrm{d}s\right]\\ &\leq \mathbb{E}\left[\|\mathbf{U}_0^N\|^p_{\ell_0^p(\mathbb{T}_N)}\right] + \mathsf{D}\frac{p(p-1)}{2}\mathbb{E}\left[\int_0^t \|\mathbf{U}^N(s)\|^{p-2}_{\ell_0^{p-2}(\mathbb{T}_N)}\mathrm{d}s\right].\end{aligned} \tag{26}$$

(ii) *There exist six positive constants $\mathsf{c}_0^{(p)}$, $\mathsf{c}_1^{(p)}$, $\mathsf{c}_2^{(p)}$, $\mathsf{d}_0^{(p)}$, $\mathsf{d}_1^{(p)}$ and $\mathsf{d}_2^{(p)}$, depending only on D, ν and p such that we have*

$$\forall t > 0, \qquad \mathbb{E}\left[\int_0^t \|\mathbf{U}^N(s)\|^p_{\ell_0^p(\mathbb{T}_N)}\mathrm{d}s\right] \leq \mathsf{c}_0^{(p)} + \mathsf{c}_1^{(p)}\mathbb{E}\left[\|\mathbf{U}_0^N\|^p_{\ell_0^p(\mathbb{T}_N)}\right] + \mathsf{c}_2^{(p)}t, \tag{27}$$

and

$$\forall t > 0, \qquad \sup_{s\in[0,t]}\mathbb{E}\left[\|\mathbf{U}^N(s)\|^p_{\ell_0^p(\mathbb{T}_N)}\right] \leq \mathsf{d}_0^{(p)} + \mathsf{d}_1^{(p)}\mathbb{E}\left[\|\mathbf{U}_0^N\|^p_{\ell_0^p(\mathbb{T}_N)}\right] + \mathsf{d}_2^{(p)}t. \tag{28}$$

The proof of Lemma 2.5 relies on the following $\ell_0^p(\mathbb{T}_N)$ extension of the Poincaré inequality (22), the proof of which is postponed to Appendix A.

Lemma 2.6 ($\ell_0^p(\mathbb{T}_N)$ Poincaré inequality)**.** *For any $\mathbf{v}\in\mathbb{R}_0^N$ and $p\in 2\mathbb{N}^*$, we have*

$$\left\langle \mathbf{D}_N^{(1,+)}\left(\mathbf{v}^{p-1}\right), \mathbf{D}_N^{(1,+)}\mathbf{v}\right\rangle_{\ell_0^2(\mathbb{T}_N)} \geq \frac{4(p-1)}{p^2}\|\mathbf{v}\|^p_{\ell_0^p(\mathbb{T}_N)}.$$

We sketch the lines of the proof of Lemma 2.5 and refer to [27, Lemma 3.16] for details.

Sketch of the proof of Lemma 2.5. Let us fix $p \in 2\mathbb{N}^*$ and apply the Itô formula to $\|\mathbf{U}^N(t)\|^p_{\ell^p_0(\mathbb{T}_N)}$. We get

$$\mathrm{d}\|\mathbf{U}^N(t)\|^p_{\ell^p_0(\mathbb{T}_N)} = p\left\langle \mathbf{U}^N(t)^{p-1}, \left(-\mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(t)) + \nu\mathbf{D}_N^{(2)}\mathbf{U}^N(t)\right)\mathrm{d}t + \mathrm{d}\mathbf{W}^{Q,N}(t)\right\rangle_{\ell^2(\mathbb{T}_N)}$$
$$+ \frac{p(p-1)}{2}\left\langle \mathbf{U}^N(t)^{p-2}, \sum_{k\geq 1}(\mathbf{g}^k)^2\right\rangle_{\ell^2(\mathbb{T}_N)} \mathrm{d}t.$$

By Lemma 2.2,

$$\left\langle \mathbf{U}^N(t)^{p-1}, -\mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(t))\right\rangle_{\ell^2(\mathbb{T}_N)} \leq 0;$$

by (20),

$$\left\langle \mathbf{U}^N(t)^{p-1}, \nu\mathbf{D}_N^{(2)}\mathbf{U}^N(t)\right\rangle_{\ell^2(\mathbb{T}_N)} = -\nu\left\langle \mathbf{D}_N^{(1,+)}\left(\mathbf{U}^N(t)^{p-1}\right), \mathbf{D}_N^{(1,+)}\mathbf{U}^N(t)\right\rangle_{\ell^2_0(\mathbb{T}_N)};$$

and by (24),

$$\left\langle \mathbf{U}^N(t)^{p-2}, \sum_{k\geq 1}(\mathbf{g}^k)^2\right\rangle_{\ell^2(\mathbb{T}_N)} \leq \mathsf{D}\|\mathbf{U}^N(t)\|^{p-2}_{\ell^{p-2}_0(\mathbb{T}_N)}.$$

Using a localisation argument to remove the stochastic integral when taking the expectation, we get (26). The estimates (27) and (28) then follow from the application of Lemma 2.6, the elementary inequality

$$\|\mathbf{U}_0^N\|^{p-2}_{\ell^{p-2}_0(\mathbb{T}_N)} \leq 1 + \|\mathbf{U}_0^N\|^p_{\ell^p_0(\mathbb{T}_N)},$$

and an inductive argument on the even values of p. □

2.2.3. *Feller property and existence of an invariant measure.* From the Lyapunov condition (25), we may also deduce the existence of an invariant measure for $(\mathbf{U}^N(t))_{t\geq 0}$, thanks to the Krylov–Bogoliubov tightness criterion [14, Corollary 3.1.2]. In order to apply the latter, it is necessary to check that $(\mathbf{U}^N(t))_{t\geq 0}$ is a Feller process, which in our case is a consequence of the following $\ell^1_0(\mathbb{T}_N)$ contraction property.

Proposition 2.7 ($\ell^1_0(\mathbb{T}_N)$ contraction for $\mathbf{U}^N$)**.** *Two solutions $(\mathbf{U}^N(t))_{t\geq 0}$ and $(\mathbf{V}^N(t))_{t\geq 0}$ of (9), driven by the same Wiener process $\mathbf{W}^{Q,N}$, with possibly different initial conditions, satisfy almost surely*

$$\forall 0 \leq s \leq t, \qquad \left\|\mathbf{U}^N(t) - \mathbf{V}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \leq \left\|\mathbf{U}^N(s) - \mathbf{V}^N(s)\right\|_{\ell^1_0(\mathbb{T}_N)}.$$

Proof. Since $(\mathbf{U}^N(t))_{t\geq 0}$ and $(\mathbf{V}^N(t))_{t\geq 0}$ are driven by the same Wiener process, then $(\mathbf{U}^N(t)-\mathbf{V}^N(t))_{t\geq 0}$ is an absolutely continuous process, and

$$\mathrm{d}(\mathbf{U}^N(t) - \mathbf{V}^N(t)) = \left(\mathbf{b}(\mathbf{U}^N(t)) - \mathbf{b}(\mathbf{V}^N(t))\right)\mathrm{d}t.$$

In particular, we can write for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{U}^N(t) - \mathbf{V}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} = \left\langle \mathbf{sign}\left(\mathbf{U}^N(t) - \mathbf{V}^N(t)\right), \mathbf{b}(\mathbf{U}^N(t)) - \mathbf{b}(\mathbf{V}^N(t))\right\rangle_{\ell^2(\mathbb{T}_N)} \leq 0,$$

where the inequality comes from Lemma 2.1.(i), and the result follows by integrating in time. □

Corollary 2.8 (Feller property for $\mathbf{U}^N$)**.** *The solution $(\mathbf{U}^N(t))_{t\geq 0}$ of Equation (9) satisfies the Feller property,* i.e. *for any continuous and bounded function $\varphi : \mathbb{R}^N_0 \to \mathbb{R}$ and any $t \geq 0$, the mapping*

$$\mathbf{u}_0 \in \mathbb{R}^N_0 \longmapsto \mathbb{E}_{\mathbf{u}_0}\left[\varphi(\mathbf{U}^N(t))\right] \in \mathbb{R}$$

is continuous and bounded, where the notation $\mathbb{E}_{\mathbf{u}_0}$ indicates that $\mathbf{U}^N(0) = \mathbf{u}_0$.

We may now state the existence result of an invariant measure.

Proposition 2.9 (Existence of an invariant measure for $\mathbf{U}^N$)**.** *The solution $(\mathbf{U}^N(t))_{t\geq 0}$ of (9) admits an invariant measure $\nu_N \in \mathcal{P}_2(\mathbb{R}^N_0)$.*

The proof is omitted because it is a standard consequence of (25), see [27, Section 3.2.2].

2.2.4. *Uniqueness of the invariant measure.* The proof of uniqueness of the invariant measure ν_N relies on the intermediary Lemmas 2.10 and 2.11 which are stated below. They follow from standard but a bit lengthy computation. For the sake of legibility of the whole argument, we therefore postpone their proofs to Appendix B.

Lemma 2.10 (Hitting any neighbourhood of 0 with positive probability)**.** *Let $(\mathbf{U}^N(t))_{t\geq 0}$ and $(\mathbf{V}^N(t))_{t\geq 0}$ be two solutions of (9) driven by the same Wiener process. Then, for all $M>0$ and all $\varepsilon>0$, there exists $t_{\varepsilon,M}>0$ such that*

$$p_{\varepsilon,M} := \inf \mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}\left(\left\|\mathbf{U}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} \leq \varepsilon\right) > 0,$$

where the notation $\mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}$ indicates that $\mathbf{U}^N(0)=\mathbf{u}_0$ and $\mathbf{V}^N(0)=\mathbf{v}_0$ and the infimum is taken over pairs of initial conditions $(\mathbf{u}_0,\mathbf{v}_0)$ such that $\|\mathbf{u}_0\|_{\ell^2_0(\mathbb{T}_N)} \vee \|\mathbf{v}_0\|_{\ell^2_0(\mathbb{T}_N)} \leq M$.

In the setting of Lemma 2.10, we let, for any $M\geq 0$,

$$\tau_M := \inf\left\{t\geq 0 : \left\|\mathbf{U}^N(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \vee \left\|\mathbf{V}^N(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \leq M\right\}. \tag{29}$$

Lemma 2.11 (Almost sure entrance in some ball)**.** *There exists $M>0$ such that for any deterministic initial conditions $\mathbf{u}_0$, $\mathbf{v}_0 \in \mathbb{R}^N_0$, $\tau_M < +\infty$ almost surely.*

We now detail how Lemmas 2.10 and 2.11 allow to complete the proof of uniqueness of an invariant measure for $\mathbf{U}^N$.

Proof of the uniqueness of an invariant measure for $\mathbf{U}^N$. We start by fixing $\varepsilon>0$, to which we associate the quantities $t_{\varepsilon,M}$ and $p_{\varepsilon,M}$ defined at Lemma 2.10, where M has been defined at Lemma 2.11. Let $(\mathbf{U}^N(t))_{t\geq 0}$ and $(\mathbf{V}^N(t))_{t\geq 0}$ start respectively from arbitrary deterministic initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$ and be driven by the same Wiener process. We define the increasing stopping time sequence

$$\begin{aligned} T_1 &:= \tau_M \\ T_2 &:= \inf\left\{t\geq T_1 + t_{\varepsilon,M} : \left\|\mathbf{U}^N(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \vee \left\|\mathbf{V}^N(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \leq M\right\} \\ T_3 &:= \inf\left\{t\geq T_2 + t_{\varepsilon,M} : \left\|\mathbf{U}^N(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \vee \left\|\mathbf{V}^N(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \leq M\right\} \\ &\vdots \end{aligned}$$

By the strong Markov property and Lemma 2.11, each term of this sequence is finite almost surely. We introduce the event

$$E_j = \left\{\left\|\mathbf{U}^N(T_j + t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(T_j + t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} > \varepsilon\right\}$$

and claim that

$$\forall J\in\mathbb{N}^*, \qquad \mathbb{P}\left(\cap_{j=1}^J E_j\right) \leq (1-p_{\varepsilon,M})^J.$$

Indeed, it is true for $J=1$ thanks to the strong Markov property and Lemma 2.10:

$$\mathbb{P}(E_1) = \mathbb{E}\left[\mathbb{P}\left(\left\|\mathbf{U}^N(\tau_M + t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(\tau_M + t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} > \varepsilon | \mathcal{F}_{\tau_M}\right)\right] \leq 1 - p_{\varepsilon,M},$$

and the general case follows by induction. Letting $J\to+\infty$, we get

$$\mathbb{P}\left(\cap_{j=1}^\infty E_j\right) = \lim_{J\to\infty} \mathbb{P}\left(\cap_{j=1}^J E_j\right) \leq \lim_{J\to\infty} (1-p_{\varepsilon,M})^J = 0,$$

and consequently, almost surely there exists some $t = T_j + t_{\varepsilon,M}$ such that

$$\|\mathbf{U}^N(t) - \mathbf{V}^N(t)\|_{\ell^1_0(\mathbb{T}_N)} \leq \|\mathbf{U}^N(t)\|_{\ell^1_0(\mathbb{T}_N)} + \|\mathbf{V}^N(t)\|_{\ell^1_0(\mathbb{T}_N)} \leq \varepsilon.$$

Now recall that thanks to Proposition 2.7, $\|\mathbf{U}^N(t) - \mathbf{V}^N(t)\|_{\ell^1_0(\mathbb{T}_N)}$ is non-increasing in time almost surely. Since ε has been chosen arbitrarily, we deduce that $\|\mathbf{U}^N(t) - \mathbf{V}^N(t)\|_{\ell^1_0(\mathbb{T}_N)}$ converges almost surely to 0 as $t\to+\infty$ when the initial conditions are deterministic, and since this assertion is true for any pair of initial conditions, it also holds for random and $\mathcal{F}_0$-measurable initial conditions. Let $\phi : \ell^1_0(\mathbb{T}_N)\to\mathbb{R}$ be a Lipschitz continuous and bounded test function, with Lipschitz constant L_ϕ. We have in particular, almost surely,

$$\lim_{t\to\infty} \left|\phi(\mathbf{U}^N(t)) - \phi(\mathbf{V}^N(t))\right| \leq \mathsf{L}_\phi \lim_{t\to\infty} \left\|\mathbf{U}^N(t) - \mathbf{V}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} = 0. \tag{30}$$

To conclude the proof, assume that there exist two invariant measures $\nu_N^{(1)}$ and $\nu_N^{(2)}$ for the solution of (9), and take random initial conditions $\mathbf{U}^N_0$ and $\mathbf{V}^N_0$ with distributions $\nu_N^{(1)}$ and $\nu_N^{(2)}$ respectively. We have for all $t\geq 0$,

$$\left|\mathbb{E}\left[\phi\left(\mathbf{U}^N_0\right)\right] - \mathbb{E}\left[\phi\left(\mathbf{V}^N_0\right)\right]\right| = \left|\mathbb{E}\left[\phi\left(\mathbf{U}^N(t)\right)\right] - \mathbb{E}\left[\phi\left(\mathbf{V}^N(t)\right)\right]\right| \leq \mathbb{E}\left[\left|\phi\left(\mathbf{U}^N(t)\right) - \phi\left(\mathbf{V}^N(t)\right)\right|\right].$$

Letting t go to $+\infty$, by (30) and the dominated convergence theorem, we have

$$\left|\mathbb{E}\left[\phi\left(\mathbf{U}^N_0\right)\right] - \mathbb{E}\left[\phi\left(\mathbf{V}^N_0\right)\right]\right| \leq \lim_{t\to\infty} \mathbb{E}\left[\left|\phi\left(\mathbf{U}^N(t)\right) - \phi\left(\mathbf{V}^N(t)\right)\right|\right] = 0.$$

As a consequence, $\mathbf{U}^N_0$ and $\mathbf{V}^N_0$ have the same distribution, meaning that $\nu_N^{(1)} = \nu_N^{(2)}$. □

Remark 2.12. This proof shows in addition that for any initial distribution, the law of $\mathbf{U}^N(t)$ converges, when $t \to +\infty$, to the invariant measure ν_N.

2.3. **The split-step scheme.** In this subsection, we first show that the implicit equation in (14) has a unique solution, which ensures the well-posedness of the sequence $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$. We then prove the existence and the uniqueness of an invariant measure $\nu_{N,\Delta t}$ for this Markov chain. The general organisation of the arguments is rather close to Subsection 2.2, and we only emphasise which points have to be adapted.

2.3.1. *Well-posedness.* The following preliminary result ensures that the scheme (14) is well-posed.

Proposition 2.13 (Well-posedness of (14))**.** *For any $\Delta t > 0$ and $\mathbf{v} \in \mathbb{R}_0^N$, there exists a unique $\mathbf{w} \in \mathbb{R}_0^N$ such that $\mathbf{w} = \mathbf{v} + \Delta t \mathbf{b}(\mathbf{w})$.*

Proof. Uniqueness. It is a straightforward consequence of Lemma 2.1.(i): if $\mathbf{w}_1$ and $\mathbf{w}_2$ are two solutions, then

$$\|\mathbf{w}_1 - \mathbf{w}_2\|_{\ell_0^1(\mathbb{T}_N)} = \langle \mathbf{sign}(\mathbf{w}_1 - \mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2\rangle_{\ell^2(\mathbb{T}_N)} = \Delta t \langle \mathbf{sign}(\mathbf{w}_1 - \mathbf{w}_2), \mathbf{b}(\mathbf{w}_1) - \mathbf{b}(\mathbf{w}_2)\rangle_{\ell^2(\mathbb{T}_N)} \leq 0.$$

Existence. The mapping $\mathbf{Id} - \Delta t \mathbf{b} : \mathbb{R}_0^N \to \mathbb{R}_0^N$ is continuous. Furthermore, by Lemmas 2.1.(ii) and (22), we have for all $\mathbf{w} \in \mathbb{R}_0^N$,

$$\frac{\langle (\mathbf{Id} - \Delta t \mathbf{b})(\mathbf{w}), \mathbf{w}\rangle_{\ell_0^2(\mathbb{T}_N)}}{\|\mathbf{w}\|_{\ell_0^2(\mathbb{T}_N)}} = \|\mathbf{w}\|_{\ell_0^2(\mathbb{T}_N)} - \Delta t \frac{\langle \mathbf{b}(\mathbf{w}), \mathbf{w}\rangle_{\ell_0^2(\mathbb{T}_N)}}{\|\mathbf{w}\|_{\ell_0^2(\mathbb{T}_N)}} \geq \|\mathbf{w}\|_{\ell_0^2(\mathbb{T}_N)} + \nu \Delta t \frac{\|\mathbf{D}_N^{(1,+)}\mathbf{w}\|^2_{\ell_0^2(\mathbb{T}_N)}}{\|\mathbf{w}\|_{\ell_0^2(\mathbb{T}_N)}} \geq (1 + \nu\Delta t)\|\mathbf{w}\|_{\ell_0^2(\mathbb{T}_N)}.$$

Thus, as a consequence of [15, Theorem 3.3], $\mathbf{Id} - \Delta t \mathbf{b}$ is surjective in $\mathbb{R}_0^N$ and, for any $\mathbf{v} \in \mathbb{R}_0^N$, there exists $\mathbf{w} \in \mathbb{R}_0^N$ such that $\mathbf{w} = \mathbf{v} + \Delta t \mathbf{b}(\mathbf{w})$. $\square$

2.3.2. *Existence of an invariant measure.* We first prove an $\ell_0^1(\mathbb{T}_N)$ contraction property in order to deduce the Feller property for $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$.

Lemma 2.14 ($\ell_0^1(\mathbb{T}_N)$ contraction for $\mathbf{U}^{N,\Delta t}$)**.** *Let $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ and $(\mathbf{V}_n^{N,\Delta t})_{n\in\mathbb{N}}$ be two solutions of (14), constructed with the same sequence of noise increments $(\Delta \mathbf{W}^{Q,N})_{n\in\mathbb{N}^*}$. Then, almost surely and for any $n \in \mathbb{N}$,*

$$\left\|\mathbf{U}_{n+1}^{N,\Delta t} - \mathbf{V}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)} \leq \left\|\mathbf{U}_n^{N,\Delta t} - \mathbf{V}_n^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}.$$

Proof. From Equations (14) and Lemma 2.1.(ii), we write

$$\begin{aligned}
&\left\|\mathbf{U}_{n+1}^{N,\Delta t} - \mathbf{V}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)} \\
&= \left\|\mathbf{U}_{n+\frac12}^{N,\Delta t} - \mathbf{V}_{n+\frac12}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)} \\
&= \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac12}^{N,\Delta t} - \mathbf{V}_{n+\frac12}^{N,\Delta t}\right), \mathbf{U}_{n+\frac12}^{N,\Delta t} - \mathbf{V}_{n+\frac12}^{N,\Delta t}\right\rangle_{\ell^2(\mathbb{T}_N)} \\
&= \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac12}^{N,\Delta t} - \mathbf{V}_{n+\frac12}^{N,\Delta t}\right), \mathbf{U}_{n}^{N,\Delta t} - \mathbf{V}_{n}^{N,\Delta t}\right\rangle_{\ell^2(\mathbb{T}_N)} + \Delta t \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac12}^{N,\Delta t} - \mathbf{V}_{n+\frac12}^{N,\Delta t}\right), \mathbf{b}\left(\mathbf{U}_{n+\frac12}^{N,\Delta t}\right) - \mathbf{b}\left(\mathbf{V}_{n+\frac12}^{N,\Delta t}\right)\right\rangle_{\ell^2(\mathbb{T}_N)} \\
&\leq \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac12}^{N,\Delta t} - \mathbf{V}_{n+\frac12}^{N,\Delta t}\right), \mathbf{U}_{n}^{N,\Delta t} - \mathbf{V}_{n}^{N,\Delta t}\right\rangle_{\ell^2(\mathbb{T}_N)} \\
&\leq \left\|\mathbf{U}_n^{N,\Delta t} - \mathbf{V}_n^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}.
\end{aligned}$$

$\square$

Remark 2.15. The choice of the split-step backward Euler scheme is essential for the $\ell_0^1(\mathbb{T}_N)$ contraction property to hold. Indeed, consider for instance two sequences $(\widetilde{\mathbf{U}}_n^{N,\Delta t})_{n\in\mathbb{N}}$ and $(\widetilde{\mathbf{V}}_n^{N,\Delta t})_{n\in\mathbb{N}}$ built via an explicit Euler method, that is,

$$\widetilde{\mathbf{U}}_{n+1}^{N,\Delta t} = \widetilde{\mathbf{U}}_n^{N,\Delta t} + \Delta t \mathbf{b}\left(\widetilde{\mathbf{U}}_n^{N,\Delta t}\right) + \Delta \mathbf{W}_{n+1}^{Q,N}$$

(and naturally, the same construction for $(\widetilde{\mathbf{V}}_n^{N,\Delta t})_{n\in\mathbb{N}}$), then the expansion of the $\ell_0^1(\mathbb{T}_N)$ distance gives

$$\begin{aligned}
\left\|\widetilde{\mathbf{U}}_{n+1}^{N,\Delta t} - \widetilde{\mathbf{V}}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)} &= \mathbf{sign}\left\langle \left(\widetilde{\mathbf{U}}_{n+1}^{N,\Delta t} - \widetilde{\mathbf{V}}_{n+1}^{N,\Delta t}\right), \widetilde{\mathbf{U}}_n^{N,\Delta t} - \widetilde{\mathbf{V}}_n^{N,\Delta t}\right\rangle_{\ell^2(\mathbb{T}_N)} \\
&\quad + \Delta t \left\langle \mathbf{sign}\left(\widetilde{\mathbf{U}}_{n+1}^{N,\Delta t} - \widetilde{\mathbf{V}}_{n+1}^{N,\Delta t}\right), \mathbf{b}\left(\widetilde{\mathbf{U}}_n^{N,\Delta t}\right) - \mathbf{b}\left(\widetilde{\mathbf{V}}_n^{N,\Delta t}\right)\right\rangle_{\ell^2(\mathbb{T}_N)}.
\end{aligned}$$

Thus, we would need to control the second term of the right-hand side in the above equation, which is delicate given that $\mathbf{b}$ is not globally Lipschitz.

As for the semi-discrete scheme, Lemma 2.14 induces the following property.

Corollary 2.16 (Feller property for $\mathbf{U}^{N,\Delta t}$)**.** *The solution $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ of (14) has the Feller property.*

The existence of an invariant measure follows again from the application of the Krylov–Bogoliubov theorem to a Lyapunov condition similar to (25).

Proposition 2.17 (Existence of an invariant measure for $\mathbf{U}^{N,\Delta t}$)**.** *For all $n \geq 1$, we have*

$$\frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{l+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq \frac{1}{2n\nu\Delta t}\left\|\mathbf{u}_0\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + \frac{\mathsf{D}}{2\nu} + \Delta t\mathsf{D}. \tag{31}$$

As a consequence, the sequence $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ admits an invariant measure $\nu_{N,\Delta t} \in \mathcal{P}_2(\mathbb{R}_0^N)$.

Proof. Starting from the first equation in (14), we have

$$\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t} - \Delta t\mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right)\right\|_{\ell_0^2(\mathbb{T}_N)}^2 = \left\|\mathbf{U}_n^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2,$$

by expanding the left-hand side, we derive the inequality

$$\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \left\|\mathbf{U}_n^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + 2\Delta t\left\langle\mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right), \mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\rangle_{\ell_0^2(\mathbb{T}_N)}.$$

Using Lemma 2.1.(ii), we get

$$\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \left\|\mathbf{U}_n^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 - 2\nu\Delta t\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2. \tag{32}$$

Now, from the second equation in (14), we have

$$\left\|\mathbf{U}_{n+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 = \left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + 2\left\langle\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}, \Delta\mathbf{W}_{n+1}^{Q,N}\right\rangle_{\ell_0^2(\mathbb{T}_N)} + \left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_{\ell_0^2(\mathbb{T}_N)}^2. \tag{33}$$

Injecting Inequality (32) into Equation (33), we get

$$\left\|\mathbf{U}_{n+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 - \left\|\mathbf{U}_n^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq -2\nu\Delta t\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + 2\left\langle\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}, \Delta\mathbf{W}_{n+1}^{Q,N}\right\rangle_{\ell_0^2(\mathbb{T}_N)} + \left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_{\ell_0^2(\mathbb{T}_N)}^2. \tag{34}$$

By definition of $\mathbf{W}^{Q,N}$ and from (24), we have

$$\mathbb{E}\left[\left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] = \Delta t\sum_{k\geq 1}\left\|\mathbf{g}^k\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \mathsf{D}\Delta t. \tag{35}$$

On the other hand, the variables $\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}$ and $\Delta\mathbf{W}_{n+1}^{Q,N}$ are independent, so that taking the expectation in (34) yields

$$\mathbb{E}\left[\left\|\mathbf{U}_{n+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] - \mathbb{E}\left[\left\|\mathbf{U}_n^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq -2\nu\Delta t\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] + \mathsf{D}\Delta t,$$

which is valid for any $n \in \mathbb{N}$, so that we get a telescopic sum:

$$\begin{aligned}\mathbb{E}\left[\left\|\mathbf{U}_n^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] - \left\|\mathbf{u}_0\right\|_{\ell_0^2(\mathbb{T}_N)}^2 &= \sum_{l=0}^{n-1}\left(\mathbb{E}\left[\left\|\mathbf{U}_{l+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] - \mathbb{E}\left[\left\|\mathbf{U}_l^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right]\right)\\ &\leq -2\nu\Delta t\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] + n\Delta t\mathsf{D}.\end{aligned}$$

Hence,

$$2\nu\Delta t\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq \left\|\mathbf{u}_0\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + n\Delta t\mathsf{D}. \tag{36}$$

Besides,

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{l+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] = \mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] + \mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right], \tag{37}$$

and by (24),

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] = \Delta t\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1,+)}\mathbf{g}^k\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \Delta t\mathsf{D}.$$

Injecting this bound into (37), and (37) into (36), we get (31).

Since $\|\mathbf{D}_N^{(1,+)}\cdot\|_{\ell_0^2(\mathbb{T}_N)}$ defines a norm on $\mathbb{R}_0^N$ and since from Corollary 2.16, the sequence $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ has the Feller property, the existence of an invariant measure $\nu_{N,\Delta t} \in \mathcal{P}_2(\mathbb{R}_0^N)$ now follows from the Krylov–Bogoliubov theorem. ☐

2.3.3. *Uniqueness of the invariant measure.* Similarly to Subsection 2.2, we first state the intermediary Lemmas 2.18 and 2.19.

Lemma 2.18 (Hitting any neighbourhood of 0 with positive probability)**.** *Let $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ and $(\mathbf{V}_n^{N,\Delta t})_{n\in\mathbb{N}}$ be two solutions of (14) constructed with the same sequence of noise increments $(\Delta\mathbf{W}_n^{Q,N})_{n\in\mathbb{N}^*}$. For any $\varepsilon>0$ and any $M>0$, there exists $n_{\varepsilon,M}\in\mathbb{N}$ such that*

$$p_{\varepsilon,M} := \inf \mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}\left(\left\|\mathbf{U}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)} + \left\|\mathbf{V}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)} \leq \varepsilon\right) > 0,$$

where the notation $\mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}$ indicates that $\mathbf{U}_0^{N,\Delta t}=\mathbf{u}_0$ and $\mathbf{V}_0^{N,\Delta t}=\mathbf{v}_0$ and the infimum is taken over pairs of initial conditions $(\mathbf{u}_0,\mathbf{v}_0)$ such that $\|\mathbf{u}_0\|_{\ell_0^2(\mathbb{T}_N)}\vee\|\mathbf{v}_0\|_{\ell_0^2(\mathbb{T}_N)}\leq M$.

In the setting of Lemma 2.18, we now define

$$\eta_M := \inf\left\{n\in\mathbb{N} : \left\|\mathbf{U}_{n+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)} \vee \left\|\mathbf{V}_{n+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)} \leq M\right\}.$$

The following lemma is the time-discrete version of Lemma 2.11. The proof is omitted as it is very similar to its time-continuous counterpart.

Lemma 2.19 (Almost sure entrance in some ball)**.** *There exists $M>0$ such that for any initial conditions $\mathbf{u}_0$, $\mathbf{v}_0\in\mathbb{R}_0^N$ for the sequences $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ and $(\mathbf{V}_n^{N,\Delta t})_{n\in\mathbb{N}}$, $\eta_M<+\infty$ almost surely.*

The proof of Lemma 2.18 is quite different from the proof of Lemma 2.10, therefore it is detailed in Appendix B. On the contrary, Lemma 2.19 essentially follows from the same arguments as Lemma 2.11 and therefore we omit its proof. Finally, given Lemmas 2.14, 2.18 and 2.19, the proof of the uniqueness of the invariant measure $\nu_{N,\Delta t}$ of the split-step scheme is an obvious adaptation of the proof for the semi-discrete scheme.

3. Convergence of invariant measures: semi-discrete scheme towards SPDE

This section is dedicated to the proof of the first statement in Theorem 1.7, namely the convergence of μ_N to μ. The general sketch of the proof is detailed in Subsection 3.1. Subsections 3.2 and 3.3 contain the proofs of the main arguments. A discussion on the rate of convergence associated with Theorem 1.7 is then provided in Subsection 3.4.

3.1. General sketch of the proof.

The first ingredient of the proof is the following series of uniform estimates on ν_N.

Proposition 3.1 (Uniform $\ell_0^p(\mathbb{T}_N)$, $h_0^1(\mathbb{T}_N)$ and $h_0^2(\mathbb{T}_N)$ estimates on ν_N)**.** *Let $\mathbf{V}^N$ be a random variable in $\mathbb{R}_0^N$ with distribution ν_N. For all $p\in[1,+\infty)$, there exists a constant $\mathsf{C}^{0,p}\in[0,+\infty)$, which does not depend on N, such that*

$$\mathbb{E}\left[\|\mathbf{V}^N\|_{\ell_0^p(\mathbb{T}_N)}^p\right]\leq\mathsf{C}^{0,p}.$$

Besides, there exist constants $\mathsf{C}^{1,2},\mathsf{C}^{2,2}\in[0,+\infty)$, which do not depend on N, such that

$$\mathbb{E}\left[\|\mathbf{D}_N^{(1,+)}\mathbf{V}^N\|_{\ell_0^2(\mathbb{T}_N)}^2\right]\leq\mathsf{C}^{1,2},\qquad\mathbb{E}\left[\|\mathbf{D}_N^{(2)}\mathbf{V}^N\|_{\ell_0^2(\mathbb{T}_N)}^2\right]\leq\mathsf{C}^{2,2}.$$

Remark 3.2 (Notation for constants)**.** Throughout this section and the next one, we use the superscript indices m,p in the notation for constants in order to refer to bounds over Sobolev $W^{m,p}$ norms.

For all $N\geq1$, we recall the definition (16) of the measure $\mu_N\in\mathcal{P}_2(L_0^2(\mathbb{T}))$. Proposition 3.1 implies the following result.

Corollary 3.3 (Relative compactness and $L_0^p(\mathbb{T})$, $H_0^1(\mathbb{T})$ and $H_0^2(\mathbb{T})$ estimates on μ_N)**.** *The sequence $(\mu_N)_{N\geq1}$ is relatively compact in $\mathcal{P}_2(L_0^2(\mathbb{T}))$, and any subsequential limit μ^* has the property that if v^* is a random variable in $L_0^2(\mathbb{T})$ with distribution μ^*, then for all $p\in[1,+\infty)$,*

$$\mathbb{E}\left[\|v\|_{L_0^p(\mathbb{T})}^p\right]\leq\mathsf{C}^{0,p},$$

and

$$\mathbb{E}\left[\|v\|_{H_0^1(\mathbb{T})}^2\right]\leq\mathsf{C}^{1,2},\qquad\mathbb{E}\left[\|v\|_{H_0^2(\mathbb{T})}^2\right]\leq\mathsf{C}^{2,2}.$$

As a consequence of Corollary 3.3, in order to prove that μ_N converges in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ to the unique invariant measure μ of (1), it suffices to show that any subsequential limit μ^* of this sequence coincides with μ. To this aim, we let μ^* be such a limit, and for convenience we still denote by $(\mu_N)_{N\geq1}$ the extracted subsequence which converges to μ^*. We then use the Skorohod representation theorem to construct, on the same probability space, a sequence of $\mathcal{F}_0$-measurable random variables $\mathbf{U}_0^N\in\mathbb{R}_0^N$ and a random variable $u_0^*\in L_0^2(\mathbb{T})$ such that:

- for all $N\geq1$, $\mathbf{U}_0^N\sim\nu_N$,
- $u_0^*\sim\mu^*$,
- the sequence $\Psi_N\mathbf{U}_0^N$ converges almost surely to u_0^* in $L_0^2(\mathbb{T})$.

Notice that by Corollary 3.3, $u_0 \in H_0^2(\mathbb{T})$, almost surely, which thus allows to take this random variable as an initial condition for (1). On this probability space, we therefore let $(W^Q(t))_{t\geq 0}$ be a Q-Wiener process, $(u^*(t))_{t\geq 0}$ be the solution to (1) with initial condition u_0^* and driven by $(W^Q(t))_{t\geq 0}$, and for all $N \geq 1$, we let $(\mathbf{U}^N(t))_{t\geq 0}$ be the solution to (9) with initial condition $\mathbf{U}_0^N \sim \nu_N$ and driven by the Wiener process $\mathbf{W}^{Q,N} = \Pi_N W^Q$. We finally denote by $u^N(t) = \Psi_N \mathbf{U}^N(t)$ the piecewise constant reconstruction of $\mathbf{U}^N(t)$ on $\mathbb{T}$.

Proposition 3.4 (Finite-time convergence of $u^N(t)$)**.** *In the setting introduced above, for all $t \geq 0$,*

$$\lim_{N\to+\infty} \mathbb{E}\left[\|u^N(t) - u^*(t)\|_{L_0^2(\mathbb{T})}^2\right] = 0.$$

Proposition 3.4 implies in particular that the law of $u^N(t)$ converges in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ to the law of $u^*(t)$. But since $u^N(t) = \Psi_N \mathbf{U}^N(t)$, and the process $(\mathbf{U}^N(t))_{t\geq 0}$ is stationary, its law does not depend on t. Therefore, the law of $u^*(t)$ does not depend on t either; in other words, μ^* is an invariant measure for (1). By the uniqueness result of Proposition 1.2, we deduce that $\mu^* = \mu$ and the proof of (17) is completed.

Remark 3.5. This argument shows that any subsequential limit μ^* of $(\mu_N)_{N\geq 1}$ is an invariant measure for (1), therefore it provides an alternative proof for the existence part in [28, Theorem 2]. The uniqueness part remains crucial to identify all subsequential limits and obtain the convergence of the sequence $(\mu_N)_{N\geq 1}$.

3.2. **Proofs of Proposition 3.1 and Corollary 3.3.** We first detail the proof of Proposition 3.1.

Proof of Proposition 3.1. The proof is divided in 4 steps. The $\ell_0^p(\mathbb{T}_N)$ estimate is derived in Step 1. An intermediary result is stated in Step 2, and the $h_0^1(\mathbb{T}_N)$ and $h_0^2(\mathbb{T}_N)$ estimates are respectively derived in Steps 3 and 4.

Step 1: $\ell_0^p(\mathbb{T}_N)$ estimate. The argument is inspired from [23, Proposition 4.24] and relies on the finite-time uniform estimates from Lemma 2.5. For any $M > 0$, $p \in 2\mathbb{N}^*$ and $t > 0$, we first write

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{V}^N\|_{\ell_0^p(\mathbb{T}_N)}^p \wedge M\right] &= \frac{1}{t}\int_0^t \int_{\mathbb{R}_0^N} \mathbb{E}_{\mathbf{u}_0}\left[\|\mathbf{U}^N(s)\|_{\ell_0^p(\mathbb{T}_N)}^p \wedge M\right] \mathrm{d}\nu_N(\mathbf{u}_0)\mathrm{d}s \\
&= \int_{\mathbb{R}_0^N} \frac{1}{t}\int_0^t \mathbb{E}_{\mathbf{u}_0}\left[\|\mathbf{U}^N(s)\|_{\ell_0^p(\mathbb{T}_N)}^p \wedge M\right] \mathrm{d}s\mathrm{d}\nu_N(\mathbf{u}_0) \\
&\leq \int_{\mathbb{R}_0^N} \left(\frac{1}{t}\int_0^t \mathbb{E}_{\mathbf{u}_0}\left[\|\mathbf{U}^N(s)\|_{\ell_0^p(\mathbb{T}_N)}^p\right] \mathrm{d}s\right) \wedge M\mathrm{d}\nu_N(\mathbf{u}_0) \\
&\leq \int_{\mathbb{R}_0^N} \left(\frac{1}{t}\mathsf{c}_0^{(p)} + \mathsf{c}_1^{(p)}\frac{\|\mathbf{u}_0\|_{\ell_0^p(\mathbb{T}_N)}^p}{t} + \mathsf{c}_2^{(p)}\right) \wedge M\mathrm{d}\nu_N(\mathbf{u}_0),
\end{aligned}$$

where we have used (27) in Lemma 2.5 at the last line. Letting $t \to +\infty$, we get from the dominated convergence theorem,

$$\mathbb{E}\left[\|\mathbf{V}^N\|_{\ell_0^p(\mathbb{T}_N)}^p \wedge M\right] \leq \int_{\mathbb{R}_0^N} \lim_{t\to\infty}\left(\frac{1}{t}\mathsf{c}_0^{(p)} + \mathsf{c}_1^{(p)}\frac{\|\mathbf{u}_0\|_{\ell_0^p(\mathbb{T}_N)}^p}{t} + \mathsf{c}_2^{(p)}\right) \wedge M\mathrm{d}\nu_N(\mathbf{u}_0) = \mathsf{c}_2^{(p)} \wedge M \leq \mathsf{c}_2^{(p)} =: \mathsf{C}^{0,p},$$

and the result for $p \in 2\mathbb{N}^*$ follows by letting $M \to +\infty$ and using the monotone convergence theorem. This result extends readily to the general case $p \in [1, +\infty)$ by using the Jensen inequality.

Step 2: intermediary estimate. Let $p \in 2\mathbb{N}^*$. Taking $\mathbf{U}_0^N = \mathbf{V}^N$ in (26), and using the result of Step 1, we obtain by stationarity

$$\nu p\mathbb{E}\left[\left\langle \mathbf{D}_N^{(1,+)}\left((\mathbf{V}^N)^{p-1}\right), \mathbf{D}_N^{(1,+)}\mathbf{V}^N\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right] \leq \mathsf{D}\frac{p(p-1)}{2}\mathbb{E}\left[\|\mathbf{V}^N\|_{\ell_0^{p-2}(\mathbb{T}_N)}^{p-2}\right] \leq \mathsf{D}\frac{p(p-1)}{2}\mathsf{C}^{0,p-2}, \tag{38}$$

with the convention that $\mathsf{C}^{0,0} = 1$.

Step 3: $h_0^1(\mathbb{T}_N)$ estimate. Take $p = 2$ in (38) to get

$$\mathbb{E}\left[\|\mathbf{D}_N^{(1,+)}\mathbf{V}^N\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq \frac{2\mathsf{D}}{\nu} =: \mathsf{C}^{1,2}.$$

Step 4: $h_0^2(\mathbb{T}_N)$ estimate. Let $(\mathbf{U}^N(t))_{t\geq 0}$ be the solution of (9) with initial distribution ν_N. By Itô's formula, for all $t \geq 0$,

$$\begin{aligned}
\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}^N(t)\right\|_{\ell_0^2(\mathbb{T}_N)}^2 &= \left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_0^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + 2\int_0^t \left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathbf{D}_N^{(1,+)}\mathbf{b}(\mathbf{U}^N(s))\right\rangle_{\ell_0^2(\mathbb{T}_N)} \mathrm{d}s \\
&\quad + 2\int_0^t \left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathrm{d}\left(\mathbf{D}_N^{(1,+)}\mathbf{W}^{Q,N}\right)(s)\right\rangle_{\ell_0^2(\mathbb{T}_N)} + t\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1,+)}\mathbf{g}^k\right\|_{\ell_0^2(\mathbb{T}_N)}^2.
\end{aligned} \tag{39}$$

The third term of the right-hand side is a martingale since

$$\sum_{k\geq 1}\mathbb{E}\left[\int_0^t \left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathbf{D}_N^{(1,+)}\mathbf{g}^k\right\rangle_{\ell_0^2(\mathbb{T}_N)}^2 \mathrm{d}s\right] \leq t\left(\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1,+)}\mathbf{g}^k\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right)\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{V}^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq t\mathsf{D}\mathsf{C}^{1,2} < +\infty, \tag{40}$$

where we have used the stationarity of $(\mathbf{U}^N(t))_{t\geq 0}$, Inequality (24), and the $h_0^1(\mathbb{T}_N)$ estimate from Step 3. Thus, taking the expectation in (39) and using the stationarity and Inequality (24) again, we get

$$-2\mathbb{E}\left[\left\langle \mathbf{D}_N^{(1,+)}\mathbf{V}^N, \mathbf{D}_N^{(1,+)}\mathbf{b}(\mathbf{V}^N)\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right] = \sum_{k\geq 1}\left\|\mathbf{D}_N^{(1,+)}\mathbf{g}^k\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \mathsf{D}.$$

For any $\mathbf{v}\in\mathbb{R}_0^N$, we have, by (20),

$$\begin{aligned}\left\langle \mathbf{D}_N^{(1,+)}\mathbf{v}, \mathbf{D}_N^{(1,+)}\mathbf{b}(\mathbf{v})\right\rangle_{\ell_0^2(\mathbb{T}_N)} &= -\left\langle \mathbf{D}_N^{(2)}\mathbf{v}, \mathbf{b}(\mathbf{v})\right\rangle_{\ell_0^2(\mathbb{T}_N)}\\ &= -\left\langle \mathbf{D}_N^{(2)}\mathbf{v}, \left(-\mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{v}) + \nu\mathbf{D}_N^{(2)}\mathbf{v}\right)\right\rangle_{\ell_0^2(\mathbb{T}_N)}\\ &= \left\langle \mathbf{D}_N^{(2)}\mathbf{v}, \mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{v})\right\rangle_{\ell_0^2(\mathbb{T}_N)} - \nu\|\mathbf{D}_N^{(2)}\mathbf{v}\|_{\ell_0^2(\mathbb{T}_N)}^2,\end{aligned}$$

so that

$$\begin{aligned}2\nu\mathbb{E}\left[\|\mathbf{D}_N^{(2)}\mathbf{V}^N\|_{\ell_0^2(\mathbb{T}_N)}^2\right] &\leq 2\mathbb{E}\left[\left\langle \mathbf{D}_N^{(2)}\mathbf{V}^N, \mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{V}^N)\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right] + \mathsf{D}\\ &\leq 2\sqrt{\mathbb{E}\left[\|\mathbf{D}_N^{(2)}\mathbf{V}^N\|_{\ell_0^2(\mathbb{T}_N)}^2\right]}\sqrt{\mathbb{E}\left[\|\mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{V}^N)\|_{\ell_0^2(\mathbb{T}_N)}^2\right]} + \mathsf{D},\end{aligned}$$

thanks to the Cauchy–Schwarz inequality. We now write

$$\begin{aligned}&\mathbb{E}\left[\|\mathbf{D}_N^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{V}^N)\|_{\ell_0^2(\mathbb{T}_N)}^2\right]\\ &= \mathbb{E}\left[N\sum_{i\in\mathbb{T}_N}\left(\bar{A}(V_i^N, V_{i+1}^N) - \bar{A}(V_{i-1}^N, V_i^N)\right)^2\right]\\ &= \mathbb{E}\left[N\sum_{i\in\mathbb{T}_N}\left(\int_{V_i^N}^{V_{i+1}^N}\partial_2\bar{A}(V_i^N, z)\mathrm{d}z + \int_{V_{i-1}^N}^{V_i^N}\partial_1\bar{A}(z, V_i^N)\mathrm{d}z\right)^2\right]\\ &\leq 2\mathbb{E}\left[N\sum_{i\in\mathbb{T}_N}(V_{i+1}^N - V_i^N)\int_{V_i^N}^{V_{i+1}^N}\partial_2\bar{A}(V_i^N, z)^2\mathrm{d}z\right] + 2\mathbb{E}\left[N\sum_{i\in\mathbb{T}_N}(V_i^N - V_{i-1}^N)\int_{V_{i-1}^N}^{V_i^N}\partial_1\bar{A}(z, V_i^N)^2\mathrm{d}z\right]\\ &\leq 4\mathsf{C}_{\bar{A}}^2\mathbb{E}\left[N\sum_{i\in\mathbb{T}_N}(V_i^N - V_{i-1}^N)\int_{V_{i-1}^N}^{V_i^N}\left(1+|z|^{\mathsf{p}_{\bar{A}}}\right)^2\mathrm{d}z\right]\\ &\leq 8\mathsf{C}_{\bar{A}}^2\left(\mathbb{E}\left[N\sum_{i\in\mathbb{T}_N}(V_i^N - V_{i-1}^N)^2\right] + \mathbb{E}\left[N\sum_{i\in\mathbb{T}_N}(V_i^N - V_{i-1}^N)\int_{V_{i-1}^N}^{V_i^N}|z|^{2\mathsf{p}_{\bar{A}}}\mathrm{d}z\right]\right)\\ &= 8\mathsf{C}_{\bar{A}}^2\left(\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{V}^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] + \frac{1}{2\mathsf{p}_{\bar{A}}+1}\mathbb{E}\left[\left\langle \mathbf{D}_N^{(1,+)}\left((\mathbf{V}^N)^{2\mathsf{p}_{\bar{A}}+1}\right), \mathbf{D}_N^{(1,+)}\mathbf{V}^N\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right]\right)\\ &\leq 8\mathsf{C}_{\bar{A}}^2\left(\mathsf{C}^{1,2} + \frac{\mathsf{D}}{2\nu}\mathsf{C}^{0,2\mathsf{p}_{\bar{A}}}\right),\end{aligned} \tag{41}$$

where we have used (11) at the third line, the Jensen inequality at the fourth line, (12) at the fifth line and Step 3 and the intermediary estimate (38) with $p = 2\mathsf{p}_{\bar{A}} + 2$ at the last line.

We therefore get

$$2\nu\mathbb{E}\left[\|\mathbf{D}_N^{(2)}\mathbf{V}^N\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq 2\sqrt{\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{V}^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right]}\sqrt{4\mathsf{C}_{\bar{A}}^2\frac{\mathsf{D}}{\nu}\left(1+\mathsf{C}^{0,2\mathsf{p}_{\bar{A}}}\right)} + \mathsf{D}.$$

Applying Young's inequality on the right-hand side, we get

$$2\nu\mathbb{E}\left[\|\mathbf{D}_N^{(2)}\mathbf{V}^N\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq \nu\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{V}^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] + 4\mathsf{C}_{\bar{A}}^2\frac{\mathsf{D}}{\nu^2}\left(1+\mathsf{C}^{0,2\mathsf{p}_{\bar{A}}}\right) + \mathsf{D},$$

which rewrites

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{V}^N\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq 4\mathsf{C}^2_{\bar{A}}\frac{\mathsf{D}}{\nu^3}\left(1+\mathsf{C}^{0,2\mathsf{p}_{\bar{A}}}\right)+\frac{\mathsf{D}}{\nu} =: \mathsf{C}^{2,2},$$

and yields the claimed estimate. □

In order to prepare the proof of Corollary 3.3, we first introduce the interpolation operators $\Psi_N^{(1)} : \mathbb{R}^N_0 \to H^1_0(\mathbb{T})$ and $\Psi_N^{(2)} : \mathbb{R}^N_0 \to H^2_0(\mathbb{T})$ defined by the condition that for any $\mathbf{v} \in \mathbb{R}^N_0$, the function $\Psi^{(1)}\mathbf{v}$ (resp. $\Psi^{(2)}\mathbf{v}$) is linear (resp. quadratic) on each cell of the mesh $\mathcal{T}_N$, and takes the value v_i at the interface x_i. It follows from elementary computations that, for any $\mathbf{v} \in \mathbb{R}^N_0$,

$$\|\Psi_N^{(1)}\mathbf{v}\|^2_{L^2_0(\mathbb{T})} = \|\mathbf{D}_N^{(1,+)}\mathbf{v}\|^2_{\ell^2_0(\mathbb{T}_N)}, \qquad \|\Psi_N^{(2)}\mathbf{v}\|^2_{L^2_0(\mathbb{T})} = \|\mathbf{D}_N^{(2)}\mathbf{v}\|^2_{\ell^2_0(\mathbb{T}_N)}, \tag{42}$$

and

$$\left\|\Psi_N^{(1)}\mathbf{v}-\Psi_N\mathbf{v}\right\|^2_{L^2_0(\mathbb{T})} = \frac{1}{3N^2}\left\|\mathbf{D}_N^{(1,+)}\mathbf{v}\right\|^2_{\ell^2_0(\mathbb{T}_N)}, \tag{43}$$

$$\left\|\Psi_N^{(2)}\mathbf{v}-\Psi_N\mathbf{v}\right\|^2_{L^2_0(\mathbb{T})} \leq \frac{3}{20N^4}\left\|\mathbf{D}_N^{(2)}\mathbf{v}\right\|^2_{\ell^2_0(\mathbb{T}_N)} + \frac{1}{2N^2}\left\|\mathbf{D}_N^{(1,+)}\mathbf{v}\right\|^2_{\ell^2_0(\mathbb{T}_N)}, \tag{44}$$

see [27, Lemma 3.30] for details.

Proof of Corollary 3.3. For $m = 1, 2$, we denote by $\mu_N^{(m)} = \nu_N \circ (\Psi_N^{(m)})^{-1}$ the pushforward measure of ν_N by the interpolation operator $\Psi_N^{(m)}$. For all $N \geq 1$, we shall also denote by $\mathbf{V}^N$ a random variable in $\mathbb{R}^N_0$ with distribution ν_N.

Step 1: tightness on $L^2_0(\mathbb{T})$. By (42), the uniform $h^1_0(\mathbb{T}_N)$ estimate of Proposition 3.1 rewrites under the form

$$\mathbb{E}\left[\|\Psi_N^{(1)}\mathbf{V}^N\|^2_{H^1_0(\mathbb{T})}\right] \leq \mathsf{C}^{1,2}.$$

Since bounded sets of $H^1_0(\mathbb{T})$ are compact in $L^2_0(\mathbb{T})$, this uniform moment estimate shows that the sequence $(\mu_N^{(1)})_{N\geq 1}$ is tight on $L^2_0(\mathbb{T})$. Therefore, by the Prokhorov theorem [4, Theorem 5.1], it is relatively compact in $\mathcal{P}(L^2_0(\mathbb{T}))$.

It is then easy to deduce from (43) and (44), combined with the uniform $h^1_0(\mathbb{T}_N)$ and $h^2_0(\mathbb{T}_N)$ estimates from Proposition 3.1, that for any increasing sequence of integers $(N_j)_{j\geq 1}$ and probability measure μ^* on $L^2_0(\mathbb{T})$, the following three statements are equivalent:

- μ_{N_j} converges weakly to μ^*;
- $\mu_{N_j}^{(1)}$ converges weakly to μ^*;
- $\mu_{N_j}^{(2)}$ converges weakly to μ^*;

so that all three sequences $(\mu_N)_{N\geq 1}$, $(\mu_N^{(1)})_{N\geq 1}$ and $(\mu_N^{(2)})_{N\geq 1}$ are relatively compact on $L^2_0(\mathbb{T})$, with the same converging subsequences.

Step 2: moment estimates. Let $(N_j)_{j\geq 1}$ and μ^* be such that μ_{N_j} converges weakly to μ^*. For any $p \in [1, +\infty)$, the function $v \mapsto \|v\|^p_{L^p_0(\mathbb{T})}$ is lower semicontinuous on $L^2_0(\mathbb{T})$. As a consequence, by the Portmanteau theorem, if v is a random variable in $L^2_0(\mathbb{T})$ with distribution μ^*, then

$$\mathbb{E}\left[\|v\|^p_{L^p_0(\mathbb{T})}\right] \leq \liminf_{j\to+\infty}\mathbb{E}\left[\|\Psi_N\mathbf{V}^N\|^p_{L^p_0(\mathbb{T})}\right] \leq \mathsf{C}^{0,p},$$

where we have used the isometry property (15) and the uniform $\ell^p_0(\mathbb{T}_N)$ estimate from Proposition 3.1. By Step 1, the sequences $\mu_{N_j}^{(1)}$ and $\mu_{N_j}^{(2)}$ also converge weakly to μ^*, so that using the lower semicontinuity of the functions $v \mapsto \|v\|^2_{H^1_0(\mathbb{T})}$ and $v \mapsto \|v\|^2_{H^2_0(\mathbb{T})}$ on $L^2_0(\mathbb{T})$, the isometry property (42) and the uniform $h^1_0(\mathbb{T}_N)$ and $h^2_0(\mathbb{T}_N)$ estimates from Proposition 3.1, we get

$$\mathbb{E}\left[\|v\|^2_{H^1_0(\mathbb{T})}\right] \leq \mathsf{C}^{1,2}, \qquad \mathbb{E}\left[\|v\|^2_{H^2_0(\mathbb{T})}\right] \leq \mathsf{C}^{2,2}.$$

Step 3: relative compactness in $\mathcal{P}_2(L^2_0(\mathbb{T}))$. It remains to prove that the weak convergence of μ_{N_j} to μ^* can be strengthened to the Wasserstein topology. To this aim it is sufficient to check that the sequence $\|\mathbf{V}^N\|^2_{\ell^2_0(\mathbb{T}_N)}$ is uniformly integrable [31, Definition 6.8.(iii)], which easily follows from the Jensen inequality (21) and the uniform $\ell^p_0(\mathbb{T}_N)$ estimate from Proposition 3.1. □

3.3. **Proof of Proposition 3.4.** The first step in the proof of Proposition 3.4 is the decomposition of the error

$$\|u^N(t) - u^*(t)\|^2_{L^2_0(\mathbb{T})} \leq 2\left(\|\Psi_N \Pi_N u^*(t) - u^*(t)\|^2_{L^2_0(\mathbb{T})} + \|u^N(t) - \Psi_N \Pi_N u^*(t)\|^2_{L^2_0(\mathbb{T})}\right). \tag{45}$$

On the one hand, for all $t \geq 0$, an elementary computation yields the estimate

$$\|\Pi_N u^*(t) - u^*(t)\|^2_{L^2_0(\mathbb{T})} \leq \frac{1}{N^2}\|u^*(t)\|^2_{H^1_0(\mathbb{T})},$$

while on the other hand, by (15) we have

$$\|u^N(t) - \Psi_N \Pi_N u^*(t)\|^2_{L^2_0(\mathbb{T})} = \|\Psi_N \mathbf{U}^N(t) - \Psi_N \Pi_N u^*(t)\|^2_{L^2_0(\mathbb{T})} = \|\mathbf{U}^N(t) - \Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}.$$

In order to estimate both terms, we rely on the following finite-time estimates on u^*.

Lemma 3.6 (Finite-time $L^p_0(\mathbb{T})$, $H^1_0(\mathbb{T})$ and $H^2_0(\mathbb{T})$ bounds on u^*)**.** *Under the assumptions of Proposition 3.4, for all $t > 0$ there exist constants $\mathsf{C}^{*,0,p}_t, \mathsf{C}^{*,1,2}_t, \mathsf{C}^{*,2,2}_t \in [0, +\infty)$, such that*

$$\sup_{s\in[0,t]} \mathbb{E}\left[\|u^*(s)\|^p_{L^p_0(\mathbb{T})}\right] \leq \mathsf{C}^{*,0,p}_t, \qquad \sup_{s\in[0,t]} \mathbb{E}\left[\|u^*(s)\|^2_{H^1_0(\mathbb{T})}\right] \leq \mathsf{C}^{*,1,2}_t, \qquad \mathbb{E}\left[\int_0^t \|u^*(s)\|^2_{H^2_0(\mathbb{T})}\mathrm{d}s\right] \leq \mathsf{C}^{*,2,2}_t.$$

The proof of Lemma 3.6 is postponed to Appendix C. From the $H^1_0(\mathbb{T})$ bound, we already get the control

$$\mathbb{E}\left[\|\Pi_N u^*(t) - u^*(t)\|^2_{L^2_0(\mathbb{T})}\right] \leq \frac{\mathsf{C}^{*,1,2}_t}{N^2}, \tag{46}$$

on the expectation of the first term in the right-hand side of (45).

In order to estimate the second term, we use the fact that both functions A and $\overline{A}$ are locally Lipschitz continuous, respectively on $\mathbb{R}$ and $\mathbb{R}^2$. In this purpose, we fix $M \geq 0$ and introduce the stopping time

$$\tau^N_{(M)} := \inf\left\{t \geq 0 : \|\mathbf{U}^N(t)\|_{\ell^\infty_0(\mathbb{T}_N)} \geq M \text{ or } \|u^*(t)\|_{L^\infty_0(\mathbb{T})} \geq M\right\}.$$

We also denote by $\mathsf{L}_{(M)}$ a Lipschitz constant of A on $[-M, M]$ and of $\overline{A}$ on $[-M, M]^2$. We now write

$$\mathbb{E}\left[\|\mathbf{U}^N(t) - \Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] = \mathbb{E}\left[\|\mathbf{U}^N(t) - \Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t\leq\tau^N_{(M)}\}}\right] + \mathbb{E}\left[\|\mathbf{U}^N(t) - \Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t>\tau^N_{(M)}\}}\right].$$

The terms of the right-hand side are respectively estimated in Lemmas 3.7 and 3.8.

Lemma 3.7 (Finite-time convergence in the Lipschitz case)**.** *Under the assumptions of Proposition 3.4, for all $t > 0$ and $\delta_1, \delta_2 > 0$ such that*

$$\delta_1\nu + \delta_2\mathsf{L}_{(M)} \leq 2\nu, \tag{47}$$

there exists a constant $\mathfrak{C}(t, \mathsf{L}_{(M)}, \delta_1, \delta_2)$ such that for any $N \geq 1$,

$$\mathbb{E}\left[\|\mathbf{U}^N(t) - \Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t\leq\tau^N_{(M)}\}}\right] \leq \mathrm{e}^{\gamma_{(M)}t}\mathbb{E}\left[\|\mathbf{U}^N_0 - \Pi_N u^*_0\|^2_{\ell^2_0(\mathbb{T}_N)}\right] + \frac{\mathfrak{C}(t, \mathsf{L}_{(M)}, \delta_1, \delta_2)}{N^2},$$

with

$$\gamma_{(M)} = -2\nu + \delta_1\nu + \left(\delta_2 + \frac{3}{\delta_2}\right)\mathsf{L}_{(M)}.$$

Proof. For all $t \geq 0$, we define the vector $\mathbf{e}^N(t) \in \mathbb{R}^N_0$ by $\mathbf{e}^N(t) = \mathbf{U}^N(t) - \Pi_N u^*(t)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}^N(t) = \mathbf{D}^{(1,-)}_N \mathbf{f}^N(t), \tag{48}$$

where, for all $i \in \mathbb{T}_N$,

$$f^N_i(t) = -\left(\overline{A}(U^N_i(t), U^N_{i+1}(t)) - A(u^*(t, x_i))\right) + \nu\left((\mathbf{D}^{(1,+)}_N \mathbf{U}^N(t))_i - \partial_x u^*(t, x_i)\right).$$

Using (19), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)} = 2\langle \mathbf{e}^N(t), \mathbf{D}^{(1,-)}_N \mathbf{f}^N(t)\rangle_{\ell^2_0(\mathbb{T}_N)} = -2\langle \mathbf{D}^{(1,+)}_N \mathbf{e}^N(t), \mathbf{f}^N(t)\rangle_{\ell^2(\mathbb{T}_N)},$$

and now proceed to control the right-hand side of this identity. In this purpose we introduce the zeroth- and first-order errors $\mathbf{r}^{N,(0)}(t)$ and $\mathbf{r}^{N,(1)}(t)$ defined by

$$r^{N,(0)}_i(t) = u^*(t, x_i) - (\Pi_N u^*(t))_i, \qquad r^{N,(1)}_i(t) = \partial_x u^*(t, x_i) - (\mathbf{D}^{(1,+)}_N \Pi_N u^*(t))_i.$$

On the one hand,

$$(\mathbf{D}^{(1,+)}_N \mathbf{U}^N(t))_i - \partial_x u^*(t, x_i) = (\mathbf{D}^{(1,+)}_N \mathbf{e}^N(t))_i - r^{N,(1)}_i(t),$$

on the other hand, for all $t \le \tau^N_{(M)}$, we get

$$\begin{aligned}\left|\overline{A}(U^N_i(t), U^N_{i+1}(t)) - A(u^*(t,x_i))\right| &\le \left|\overline{A}(U^N_i(t), U^N_{i+1}(t)) - \overline{A}(U^N_i(t), U^N_i(t))\right| + \left|A(U^N_i(t)) - A(u^*(t,x_i))\right| \\ &\le \mathsf{L}_{(M)}\left(\left|U^N_{i+1}(t) - U^N_i(t)\right| + \left|U^N_i(t) - u^*(t,x_i)\right|\right) \\ &\le \mathsf{L}_{(M)}\left(\left|U^N_{i+1}(t) - U^N_i(t)\right| + |e^N_i(t)| + \left|r^{N,(0)}_i(t)\right|\right).\end{aligned}$$

By the Young and Jensen inequalities, we thus deduce that, for any $\delta_1, \delta_2 > 0$,

$$\begin{aligned}-2\langle \mathbf{D}^{(1,+)}_N \mathbf{e}^N(t), \mathbf{f}^N(t)\rangle_{\ell^2(\mathbb{T}_N)} &\le -2\nu \|\mathbf{D}^{(1,+)}_N \mathbf{e}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)} + 2\nu \langle \mathbf{D}^{(1,+)}_N \mathbf{e}^N(t), \mathbf{r}^{N,(1)}(t)\rangle_{\ell^2(\mathbb{T}_N)} \\ &\quad + \frac{2\mathsf{L}_{(M)}}{N} \sum_{i\in\mathbb{T}_N} N|e^N_{i+1}(t) - e^N_i(t)| \left(\left|U^N_{i+1}(t) - U^N_i(t)\right| + |e^N_i(t)| + \left|r^{N,(0)}_i(t)\right|\right) \\ &\le \left(-2\nu + \delta_1\nu + \delta_2 \mathsf{L}_{(M)}\right) \|\mathbf{D}^{(1,+)}_N \mathbf{e}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)} + \frac{\nu}{\delta_1} \|\mathbf{r}^{N,(1)}(t)\|^2_{\ell^2(\mathbb{T}_N)} \\ &\quad + \frac{3\mathsf{L}_{(M)}}{\delta_2}\left(\frac{\|\mathbf{D}^{(1,+)}_N \mathbf{U}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)}}{N^2} + \|\mathbf{e}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)} + \|\mathbf{r}^{N,(0)}(t)\|^2_{\ell^2(\mathbb{T}_N)}\right),\end{aligned}$$

and if δ_1 and δ_2 are small enough for the inequality (47) to hold, then the discrete Poincaré inequality (22) finally yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)} \le \gamma_{(M)} \|\mathbf{e}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)} + \mathfrak{c}^N(t),$$

with

$$\mathfrak{c}^N(t) := \frac{\nu}{\delta_1}\|\mathbf{r}^{N,(1)}(t)\|^2_{\ell^2(\mathbb{T}_N)} + \frac{3\mathsf{L}_{(M)}}{\delta_2}\left(\frac{\|\mathbf{D}^{(1,+)}_N \mathbf{U}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)}}{N^2} + \|\mathbf{r}^{N,(0)}(t)\|^2_{\ell^2(\mathbb{T}_N)}\right).$$

We deduce from Grönwall's lemma that, for all $t \le \tau^N_{(M)}$,

$$\|\mathbf{e}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)} \le \mathrm{e}^{\gamma_{(M)}t}\|\mathbf{e}^N(0)\|^2_{\ell^2_0(\mathbb{T}_N)} + \int_0^t \mathrm{e}^{\gamma_{(M)}(t-s)}\mathfrak{c}^N(s)\mathrm{d}s,$$

and therefore, for all $t \ge 0$,

$$\mathbb{E}\left[\|\mathbf{e}^N(t)\|^2_{\ell^2_0(\mathbb{T}_N)} \mathbb{1}_{\{t\le\tau^N_{(M)}\}}\right] \le \mathrm{e}^{\gamma_{(M)}t}\mathbb{E}\left[\|\mathbf{e}^N(0)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] + \int_0^t \mathrm{e}^{\gamma_{(M)}(t-s)}\mathbb{E}\left[\mathfrak{c}^N(s)\right]\mathrm{d}s.$$

For all $s \in [0,t]$, the expectation of $\mathfrak{c}^N(s)$ rewrites

$$\mathbb{E}\left[\mathfrak{c}^N(s)\right] = \frac{\nu}{\delta_1}\mathbb{E}\left[\|\mathbf{r}^{N,(1)}(s)\|^2_{\ell^2(\mathbb{T}_N)}\right] + \frac{3\mathsf{L}_{(M)}}{\delta_2}\left(\frac{\mathbb{E}\left[\|\mathbf{D}^{(1,+)}_N \mathbf{U}^N(s)\|^2_{\ell^2_0(\mathbb{T}_N)}\right]}{N^2} + \mathbb{E}\left[\|\mathbf{r}^{N,(0)}(s)\|^2_{\ell^2(\mathbb{T}_N)}\right]\right).$$

On the one hand, the stationarity of $\mathbf{U}^N$ and Proposition 3.1 yield

$$\mathbb{E}\left[\|\mathbf{D}^{(1,+)}_N \mathbf{U}^N(s)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \le \mathsf{C}^{1,2}.$$

On the other hand, it easily follows from the Taylor formula that

$$\mathbb{E}\left[\|\mathbf{r}^{N,(0)}(s)\|^2_{\ell^2(\mathbb{T}_N)}\right] \le \frac{1}{N^2}\mathbb{E}\left[\|u^*(s)\|^2_{H^1_0(\mathbb{T})}\right], \qquad \mathbb{E}\left[\|\mathbf{r}^{N,(1)}(s)\|^2_{\ell^2(\mathbb{T}_N)}\right] \le \frac{4}{N^2}\mathbb{E}\left[\|u^*(s)\|^2_{H^2_0(\mathbb{T})}\right],$$

so that we get

$$\int_0^t \mathrm{e}^{\gamma_{(M)}(t-s)}\mathbb{E}\left[\mathfrak{c}^N(s)\right]\mathrm{d}s \le \frac{\mathfrak{C}(t, \mathsf{L}_{(M)}, \delta_1, \delta_2)}{N^2},$$

with

$$\mathfrak{C}(t, \mathsf{L}_{(M)}, \delta_1, \delta_2) := \int_0^t \mathrm{e}^{\gamma_{(M)}(t-s)}\left(\frac{4\nu}{\delta_1}\mathbb{E}\left[\|u^*(s)\|^2_{H^2_0(\mathbb{T})}\right] + \frac{3\mathsf{L}_{(M)}}{\delta_2}\left(\mathsf{C}^{1,2} + \mathbb{E}\left[\|u^*(s)\|^2_{H^1_0(\mathbb{T})}\right]\right)\right)\mathrm{d}s,$$

which is finite by Lemma 3.6. □

Lemma 3.8 (Uniform control over $\tau^N_{(M)}$)**.** *Under the assumptions of Proposition 3.4, for all $t > 0$ we have*

$$\lim_{M\to+\infty}\limsup_{N\to+\infty}\mathbb{E}\left[\|\mathbf{U}^N(t) - \Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t>\tau^N_{(M)}\}}\right] = 0.$$

The proof of Lemma 3.8 relies on the following result, the proof of which is postponed to Appendix C.

Lemma 3.9 (Finite-time uniform $h^1_0(\mathbb{T}_N)$ bound on $\mathbf{U}^N$)**.** *Under the assumptions of Proposition 3.4, for all $t > 0$ there exists a constant $\mathsf{S}^{1,2}_t \in [0, +\infty)$, which does not depend on N, such that*

$$\mathbb{E}\left[\sup_{s\in[0,t]} \|\mathbf{D}^{(1,+)}_N \mathbf{U}^N(s)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathsf{S}^{1,2}_t.$$

Proof of Lemma 3.8. By the Cauchy–Schwarz, triangle and Jensen inequalities, we first write

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbf{U}^N(t) - \Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)} \mathbb{1}_{\{t>\tau^N_{(M)}\}}\right] \\
&\leq \sqrt{\mathbb{E}\left[\|\mathbf{U}^N(t) - \Pi_N u^*(t)\|^4_{\ell^2_0(\mathbb{T}_N)}\right]}\sqrt{\mathbb{P}\left(t > \tau^N_{(M)}\right)} \\
&\leq \sqrt{8\left(\mathbb{E}\left[\|\mathbf{U}^N(t)\|^4_{\ell^2_0(\mathbb{T}_N)}\right] + \mathbb{E}\left[\|\Pi_N u^*(t)\|^4_{\ell^2_0(\mathbb{T}_N)}\right]\right)}\sqrt{\mathbb{P}\left(\sup_{s\in[0,t]} \|\mathbf{U}^N(s)\|_{\ell^\infty_0(\mathbb{T}_N)} \geq M \text{ or } \sup_{s\in[0,t]} \|u^*(s)\|_{L^\infty_0(\mathbb{T})} \geq M\right)}.
\end{aligned}$$

On the one hand, we deduce from (21), (7), Proposition 3.1 and Lemma 3.6 that

$$\mathbb{E}\left[\|\mathbf{U}^N(t)\|^4_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathsf{C}^{0,4}, \qquad \mathbb{E}\left[\|\Pi_N u^*(t)\|^4_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathsf{C}^{*,0,4}_t.$$

On the other hand, we get from the union bound and the Markov inequality that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\in[0,t]} \|\mathbf{U}^N(s)\|_{\ell^\infty_0(\mathbb{T}_N)} \geq M \text{ or } \sup_{s\in[0,t]} \|u^*(s)\|_{L^\infty_0(\mathbb{T})} \geq M\right) \\
&\leq \mathbb{P}\left(\sup_{s\in[0,t]} \|\mathbf{U}^N(s)\|_{\ell^\infty_0(\mathbb{T}_N)} \geq M\right) + \mathbb{P}\left(\sup_{s\in[0,t]} \|u^*(s)\|_{L^\infty_0(\mathbb{T})} \geq M\right) \\
&\leq \frac{1}{M}\mathbb{E}\left[\sup_{s\in[0,t]} \|\mathbf{U}^N(s)\|_{\ell^\infty_0(\mathbb{T}_N)}\right] + \mathbb{P}\left(\sup_{s\in[0,t]} \|u^*(s)\|_{L^\infty_0(\mathbb{T})} \geq M\right).
\end{aligned}$$

Using (23), (21), the Jensen inequality, and Lemma 3.9, we get

$$\mathbb{E}\left[\sup_{s\in[0,t]} \|\mathbf{U}^N(s)\|_{\ell^\infty_0(\mathbb{T}_N)}\right] \leq \sqrt{\mathbb{E}\left[\sup_{s\in[0,t]} \|\mathbf{D}^{(1,+)}_N \mathbf{U}^N(s)\|^2_{\ell^2_0(\mathbb{T}_N)}\right]} \leq \sqrt{\mathsf{S}^{1,2}_t},$$

while by (2) and (3),

$$\mathbb{P}\left(\sup_{s\in[0,t]} \|u^*(s)\|_{L^\infty_0(\mathbb{T})} \geq M\right) \leq \mathbb{P}\left(\sup_{s\in[0,t]} \|u^*(s)\|_{H^2_0(\mathbb{T})} \geq M\right),$$

and since u^* is a continuous $H^2_0(\mathbb{T})$-valued process, the right-hand side (which does not depend on N) converges to 0 when $M \to +\infty$. This completes the proof. ☐

We are now ready to complete the proof of Proposition 3.4.

Proof of Proposition 3.4. Combining (45) with (46) and the results of Lemmas 3.7 and 3.8, we see that in order to complete the proof of Proposition 3.4, it only remains to check that

$$\lim_{n\to+\infty} \mathbb{E}\left[\|\mathbf{U}^N_0 - \Pi_N u^*_0\|^2_{\ell^2_0(\mathbb{T}_N)}\right] = 0.$$

Using the obvious identity $\mathbf{v} = \Pi_N \Psi_N \mathbf{v}$ and (7), we first write

$$\|\mathbf{U}^N_0 - \Pi_N u^*_0\|^2_{\ell^2_0(\mathbb{T}_N)} = \|\Pi_N \Psi_N \mathbf{U}^N_0 - \Pi_N u^*_0\|^2_{\ell^2_0(\mathbb{T}_N)} \leq \|\Psi_N \mathbf{U}^N_0 - u^*_0\|^2_{L^2_0(\mathbb{T})}.$$

By the construction of $\mathbf{U}^N_0$ and u^*_0, we know that $\|\Psi_N \mathbf{U}^N_0 - u^*_0\|^2_{L^2_0(\mathbb{T})}$ converges to 0 almost surely. It remains to check that the expectation of this random variable also converges to 0, which easily follows from the moment estimates of Proposition 3.1 and Corollary 3.3 by uniform integrability. ☐

3.4. **Discussion of the rate of convergence.** The estimates (45) and (46), together with the result of Lemma 3.7, display error terms of order $1/N^2$ for the estimation of $\|u^N(t)-u^*(t)\|^2_{\ell^2_0(\mathbb{T})}$, but the localisation procedure by the stopping time $\tau^N_{(M)}$, which is used to handle the fact that A and $\overline{A}$ are not globally Lipschitz continuous, generally prevents this computation from providing a global rate of convergence.

However, under the assumption that both A and $\overline{A}$ are globally L-Lipschitz continuous, for some $\mathsf{L} \in [0,+\infty)$, a strong error estimate can be derived. Indeed, in this case, let us define on a same probability space:

- $(\mathbf{U}^N_0)_{N\geq 1}$ a sequence of $\mathcal{F}_0$-measurable random vectors such that $\mathbf{U}^N_0 \sim \nu_N$;
- u_0 a $\mathcal{F}_0$-measurable random variable in $H^2_0(\mathbb{T})$ with distribution μ;
- W^Q a Q-Wiener process;

and denote by $(\mathbf{U}^N(t))_{t\geq 0}$ and $(u(t))_{t\geq 0}$ the respective solutions to (9) and (1), with respective initial conditions $\mathbf{U}^N_0$ and u_0, and respectively driven by W^Q and $\mathbf{W}^{Q,N} := \Pi_N W^Q$. Notice that, in contrast with the setting of the previous subsections, the sequence $\Psi_N \mathbf{U}^N_0$ is not assumed to converge almost surely to u_0; on the other hand, we now know that both processes $(\mathbf{U}^N(t))_{t\geq 0}$ and $(u(t))_{t\geq 0}$ are stationary.

As a consequence, (45), (46) and Corollary 3.3 already yield

$$\mathbb{E}\left[\|u^N(t)-u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq 2\left(\frac{\mathsf{C}^{1,2}}{N^2} + \mathbb{E}\left[\|\mathbf{U}^N(t)-\Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\right).$$

One may then apply Lemma 3.7 with $\mathsf{L}_{(M)} = \mathsf{L}$ to obtain

$$\mathbb{E}\left[\|\mathbf{U}^N(t)-\Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t\leq \tau^N_{(M)}\}}\right] \leq \mathrm{e}^{\gamma t}\mathbb{E}\left[\|\mathbf{U}^N_0-\Pi_N u^*_0\|^2_{\ell^2_0(\mathbb{T}_N)}\right] + \frac{\mathfrak{C}(t,\mathsf{L},\delta_1,\delta_2)}{N^2},$$

with $\gamma = -2\nu + \delta_1\nu + (\delta_2 + \frac{3}{\delta_2})\mathsf{L}$. Since the right-hand side does not depend on M, we may take the $M \to +\infty$ limit first and thus obtain

$$\mathbb{E}\left[\|\mathbf{U}^N(t)-\Pi_N u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathrm{e}^{\gamma t}\mathbb{E}\left[\|\mathbf{U}^N_0-\Pi_N u^*_0\|^2_{\ell^2_0(\mathbb{T}_N)}\right] + \frac{\mathfrak{C}(t,\mathsf{L},\delta_1,\delta_2)}{N^2}.$$

Besides, by stationarity of $(u(t))_{t\geq 0}$ and Corollary 3.3, the constant $\mathfrak{C}(t,\mathsf{L},\delta_1,\delta_2)$ from Lemma 3.7 is bounded from above by

$$\mathfrak{C}'(t,\mathsf{L},\delta_1,\delta_2) = \int_0^t \mathrm{e}^{\gamma(t-s)}\left(\frac{4\nu}{\delta_1}\mathsf{C}^{2,2} + \frac{6\mathsf{L}}{\delta_2}\mathsf{C}^{1,2}\right)\mathrm{d}s = \left(\frac{4\nu}{\delta_1}\mathsf{C}^{2,2} + \frac{6\mathsf{L}}{\delta_2}\mathsf{C}^{1,2}\right)\int_0^t \mathrm{e}^{\gamma s}\mathrm{d}s.$$

We therefore deduce the finite-time estimate

$$\mathbb{E}\left[\|u^N(t)-u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq 2\left(\mathrm{e}^{\gamma t}\mathbb{E}\left[\|\mathbf{U}^N_0-\Pi_N u^*_0\|^2_{\ell^2_0(\mathbb{T}_N)}\right] + \frac{\mathsf{C}^{1,2}+\mathfrak{C}'(t,\mathsf{L},\delta_1,\delta_2)}{N^2}\right), \tag{49}$$

with a squared error term of order $1/N^2$. It is remarkable that this is the same order as for *deterministic* conservation laws (see for instance [21, Theorem 17.1]), so that, in our setting, the noise is sufficiently smooth in space not to deteriorate the strong error. This is in contrast with classical results for SPDEs with space-time white noise [22].

When the constant L is small enough, then it is possible to choose δ_1 and δ_2 which satisfy (47) and such that $\gamma < 0$. In this 'perturbative' case (which presents similar features to the one briefly discussed in the introduction of [9]), one may take the $t \to +\infty$ limit in (49) and obtain the long time estimate

$$\limsup_{t\to+\infty}\mathbb{E}\left[\|u^N(t)-u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \frac{\mathfrak{C}''}{N^2}, \qquad \mathfrak{C}'' = 2\left(\mathsf{C}^{1,2} + \frac{1}{-\gamma}\left(\frac{4\nu}{\delta_1}\mathsf{C}^{2,2} + \frac{6\mathsf{L}}{\delta_2}\mathsf{C}^{1,2}\right)\right).$$

Since for any $t \geq 0$, $u^N(t) \sim \mu_N$ and $u(t) \sim \mu$, we have

$$W_2(\mu_N,\mu) \leq \sqrt{\mathbb{E}\left[\|u^N(t)-u^*(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right]},$$

therefore the estimate above yields the quantitative bound

$$W_2(\mu_N,\mu) \leq \frac{\sqrt{\mathfrak{C}''}}{N}.$$

We shall check in Section 5 that the order $1/N$ for the Wasserstein distance is sharp in the case where $A = 0$.

4. Convergence of invariant measures: split-step scheme towards semi-discrete scheme

This section is dedicated to the proof of the second statement in Theorem 1.7, namely the convergence of $\nu_{N,\Delta t}$ to ν_N when $\Delta t \to 0$. We follow the same outline as in Section 3, and we only emphasise the differences in the arguments. The main results of tightness and finite-time convergence are stated in Subsection 4.1 and proved in Subsection 4.2 and 4.3, respectively. Rates of convergence are then discussed in Subsection 4.4.

Throughout this section, we fix a maximal time step $\Delta t_{\max} > 0$ and we always take $\Delta t \in (0, \Delta t_{\max}]$.

4.1. **Statement of the main arguments.** The next two results play the respective roles of Proposition 3.1 and Corollary 3.3.

Proposition 4.1 (Uniform $\ell^4_0(\mathbb{T}_N)$ and $h^1_0(\mathbb{T}_N)$ estimates on $\nu_{N,\Delta t}$)**.** *Let $\mathbf{V}^{N,\Delta t}$ be a random variable in $\mathbb{R}^N_0$ with distribution $\nu_{N,\Delta t}$, and let $\mathbf{V}^{N,\Delta t}_{\frac{1}{2}}$ be defined by*

$$\mathbf{V}^{N,\Delta t}_{\frac{1}{2}} = \mathbf{V}^{N,\Delta t} + \Delta t \mathbf{b}\left(\mathbf{V}^{N,\Delta t}_{\frac{1}{2}}\right).$$

There exist constants $\mathsf{C}^{\Delta,0,4}, \mathsf{C}^{\Delta,1,2}, \mathsf{C}^{\Delta,1,2}_{\frac{1}{2}} \in [0,+\infty)$, which only depend on ν, D and $\Delta t_{\max}$, such that

$$\mathbb{E}\left[\|\mathbf{V}^{N,\Delta t}\|^4_{\ell^4_0(\mathbb{T}_N)}\right] \leq \mathsf{C}^{\Delta,0,4}, \qquad \mathbb{E}\left[\|\mathbf{D}^{(1,+)}_N \mathbf{V}^{N,\Delta t}\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathsf{C}^{\Delta,1,2}, \qquad \mathbb{E}\left[\|\mathbf{D}^{(1,+)}_N \mathbf{V}^{N,\Delta t}_{\frac{1}{2}}\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathsf{C}^{\Delta,1,2}_{\frac{1}{2}}.$$

Corollary 4.2 (Relative compactness and $\ell^4_0(\mathbb{T}_N)$ and $h^1_0(\mathbb{T}_N)$ estimates on ν^*_N)**.** *The family of probability measures $\{\nu_{N,\Delta t}, \Delta t \in (0,\Delta t_{\max}]\}$ is relatively compact in $\mathcal{P}_2(\mathbb{R}^N_0)$, and any subsequential limit ν^*_N (when $\Delta t \to 0$) has the property that if $\mathbf{V}^{N,*}$ is a random variable in $\mathbb{R}^N_0$ with distribution ν^*_N, then*

$$\mathbb{E}\left[\|\mathbf{V}^{N,*}\|^4_{\ell^4_0(\mathbb{T}_N)}\right] \leq \mathsf{C}^{\Delta,0,4}, \qquad \mathbb{E}\left[\|\mathbf{D}^{(1,+)}_N \mathbf{V}^{N,*}\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathsf{C}^{\Delta,1,2}.$$

The proof of Proposition 4.1 is detailed in Subsection 4.2. Corollary 4.2 is straightforward.

We now let $(\Delta t_j)_{j\geq 1} \subset (0,\Delta t_{\max}]$ be a sequence of time steps decreasing to 0 and such that $\nu_{N,\Delta t_j}$ converges to some probability measure ν^*_N in $\mathcal{P}_2(\mathbb{R}^N_0)$. We use the Skorohod representation theorem to construct, on the same probability space, a sequence of $\mathcal{F}_0$-measurable random vectors $(\mathbf{U}^{N,\Delta t_j}_0)_{j\geq 1}$ and a random vector $\mathbf{U}^{N,*}_0$ such that:

- for all $j \geq 1$, $\mathbf{U}^{N,\Delta t_j}_0 \sim \nu_{N,\Delta t_j}$,
- $\mathbf{U}^{N,*}_0 \sim \nu^*_N$,
- $\mathbf{U}^{N,\Delta t_j}_0$ converges almost surely to $\mathbf{U}^{N,*}_0$.

On this probability space, we let $\mathbf{W}^{Q,N}$ be a Wiener process with covariance given by (8), we define the sequence $(\Delta \mathbf{W}^{Q,N}_n)_{n\in\mathbb{N}^*}$ by (13), and we denote by $(\mathbf{U}^{N,\Delta t_j}_n)_{n\in\mathbb{N}}$ and $(\mathbf{U}^{N,*}(t))_{t\geq 0}$ the respective solutions to (14) and (9) with initial conditions $\mathbf{U}^{N,\Delta t_j}_0$ and $\mathbf{U}^{N,*}_0$, and noises $(\Delta \mathbf{W}^{Q,N}_n)_{n\in\mathbb{N}^*}$ and $\mathbf{W}^{Q,N}$. We finally define the continuous-time piecewise constant interpolation of the split-step scheme $(\overline{\mathbf{U}}^{N,\Delta t_j}(t))_{t\geq 0}$ by

$$\forall n \in \mathbb{N}, \quad \forall t \in [n\Delta t_j, (n+1)\Delta t_j), \qquad \overline{\mathbf{U}}^{N,\Delta t_j}(t) = \mathbf{U}^{N,\Delta t_j}_n.$$

Notice that for any $t \geq 0$, $\overline{\mathbf{U}}^{N,\Delta t_j}(t) \sim \nu_{N,\Delta t_j}$.

The finite-time convergence statement reads as follows. It is proved in Subsection 4.3.

Proposition 4.3 (Finite-time convergence of $\overline{\mathbf{U}}^{N,\Delta t_j}(t)$)**.** *In the setting described above, for all $t \geq 0$,*

$$\lim_{j\to+\infty} \mathbb{E}\left[\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] = 0.$$

We deduce that the measure ν^*_N is invariant for the SDE (9), which by Theorem 1.5 completes the proof of (18) in Theorem 1.7.

4.2. **Proof of Proposition 4.1.** We first prove the $h^1_0(\mathbb{T}_N)$ estimates. Set $\mathbf{u}_0 = \mathbf{V}^{N,\Delta t}$ in the proof of Proposition 2.17. Since Theorem 1.5 asserts that $\nu_{N,\Delta t} \in \mathcal{P}_2(\mathbb{R}^N_0)$, one may take the expectation and the $n \to +\infty$ limit in (36) to get

$$\mathbb{E}\left[\|\mathbf{D}^{(1,+)}_N \mathbf{V}^{N,\Delta t}_{\frac{1}{2}}\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \frac{\mathsf{D}}{2\nu} =: \mathsf{C}^{\Delta,1,2}_{\frac{1}{2}}.$$

The same operations in (31) yield

$$\mathbb{E}\left[\|\mathbf{D}^{(1,+)}_N \mathbf{V}^{N,\Delta t}\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathsf{D}\left(\frac{1}{2\nu} + \Delta t_{\max}\right) =: \mathsf{C}^{\Delta,1,2}.$$

We now focus on the $\ell^4_0(\mathbb{T})$ estimate. Let $(\mathbf{U}^{N,\Delta t}_n)_{n\in\mathbb{N}} = (U^{N,\Delta t}_{1,n},\dots,U^{N,\Delta t}_{N,n})_{n\in\mathbb{N}}$ be a solution of (14) with a deterministic initial condition $\mathbf{u}_0$. By convexity of the function $v \mapsto v^4$, for any $\alpha, \beta \in \mathbb{R}$, we have $(\alpha-\beta)^4 \geq \alpha^4 - 4\alpha^3\beta$. In particular, for any $i \in \mathbb{T}_N$, taking $\alpha = U^{N,\Delta t}_{i,n+\frac{1}{2}}$ and $\beta = \Delta t b_i(\mathbf{U}^{N,\Delta t}_{n+\frac{1}{2}})$, we have

$$(U^{N,\Delta t}_{i,n})^4 = \left(U^{N,\Delta t}_{i,n+\frac{1}{2}} - \Delta t b_i\left(\mathbf{U}^{N,\Delta t}_{n+\frac{1}{2}}\right)\right)^4 \geq (U^{N,\Delta t}_{i,n+\frac{1}{2}})^4 - 4(U^{N,\Delta t}_{i,n+\frac{1}{2}})^3 \Delta t b_i\left(\mathbf{U}^{N,\Delta t}_{n+\frac{1}{2}}\right).$$

Hence, expanding the function $\mathbf{b}$ and summing over i, we get

$$\|\mathbf{U}^{N,\Delta t}_n\|^4_{\ell^4_0(\mathbb{T}_N)} \geq \left\|\mathbf{U}^{N,\Delta t}_{n+\frac{1}{2}}\right\|^4_{\ell^4_0(\mathbb{T}_N)} + 4\Delta t\left\langle (\mathbf{U}^{N,\Delta t}_{i,n+\frac{1}{2}})^3, \mathbf{D}^{(1,-)}_N \overline{\mathbf{A}}^N(\mathbf{U}^{N,\Delta t}_{n+\frac{1}{2}})\right\rangle_{\ell^2(\mathbb{T}_N)} - 4\nu\Delta t\left\langle (\mathbf{U}^{N,\Delta t}_{i,n+\frac{1}{2}})^3, \mathbf{D}^{(2)}_N \mathbf{U}^{N,\Delta t}_{n+\frac{1}{2}}\right\rangle_{\ell^2(\mathbb{T}_N)}.$$

We know thanks to Lemma 2.2 that the second term of the right-hand side is non-negative. Using (20) in the third term, we get

$$\|\mathbf{U}_n^{N,\Delta t}\|^4_{\ell_0^4(\mathbb{T}_N)} \geq \left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)} + 4\nu\Delta t \left\langle \mathbf{D}_N^{(1,+)}(\mathbf{U}^{N,\Delta t})^3_{n+\frac{1}{2}}, \mathbf{D}_N^{(1,+)}\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\rangle_{\ell_0^2(\mathbb{T}_N)}.$$

From Lemma 2.6, we get

$$\|\mathbf{U}_n^{N,\Delta t}\|^4_{\ell_0^4(\mathbb{T}_N)} \geq \left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)} + 3\nu\Delta t \left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}. \tag{50}$$

On the other hand, let us look at the second step of the scheme (14). By the construction of the split-step scheme, the random variables $U_{i,n+\frac{1}{2}}^{N,\Delta t}$ and $\Delta W_{i,n+1}^{Q,N}$ are independent. Since $\Delta W_{i,n+1}^{Q,N} \sim \mathcal{N}(0, \Delta t \sum_{k\geq 1}(g_i^k)^2)$ and $\sum_{k\geq 1}(g_i^k)^2 \leq \mathsf{D}$ by (24), we write

$$\begin{aligned}
\mathbb{E}\left[\left\|\mathbf{U}_{n+1}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] &= \mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t} + \Delta\mathbf{W}_{n+1}^{Q,N}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] \\
&= \mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] + \frac{6}{N}\mathbb{E}\left[\sum_{i\in\mathbb{T}_N}(U_{i,n+\frac{1}{2}}^{N,\Delta t})^2\left(\Delta W_{i,n+1}^{Q,N}\right)^2\right] + \mathbb{E}\left[\left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] \\
&\leq \mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] + 6\mathsf{D}\Delta t\mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right] + 3\mathsf{D}^2\Delta t^2.
\end{aligned} \tag{51}$$

Combining Inequalities (50) and (51), we get

$$\mathbb{E}\left[\|\mathbf{U}_n^{N,\Delta t}\|^4_{\ell_0^4(\mathbb{T}_N)}\right] \geq \mathbb{E}\left[\left\|\mathbf{U}_{n+1}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] - 6\mathsf{D}\Delta t\mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right] - 3\mathsf{D}^2\Delta t^2 + 3\nu\Delta t\mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right]$$

from which we get the telescopic sum

$$3\nu\Delta t\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] \leq \|\mathbf{u}_0\|^4_{\ell_0^4(\mathbb{T}_N)} - \mathbb{E}\left[\|\mathbf{U}_n^{N,\Delta t}\|^4_{\ell_0^4(\mathbb{T}_N)}\right] + 6\mathsf{D}\Delta t\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right] + 3n\mathsf{D}^2\Delta t^2.$$

Thus,

$$\frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] \leq \frac{1}{3\nu\Delta tn}\|\mathbf{u}_0\|^4_{\ell_0^4(\mathbb{T}_N)} + \frac{2\mathsf{D}}{\nu n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right] + \frac{\mathsf{D}^2\Delta t}{\nu}. \tag{52}$$

Recall that from (22) and Equation (36), we have

$$\frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right] \leq \frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right] \leq \frac{\|\mathbf{u}_0\|^2_{\ell_0^2(\mathbb{T}_N)}}{2\nu n\Delta t} + \frac{\mathsf{D}}{2\nu}. \tag{53}$$

Injecting (53) into (52), we get

$$\frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] \leq \frac{1}{3\nu\Delta tn}\|\mathbf{u}_0\|^4_{\ell_0^4(\mathbb{T}_N)} + \frac{2\mathsf{D}}{\nu}\left(\frac{\|\mathbf{u}_0\|^2_{\ell_0^2(\mathbb{T}_N)}}{2\nu n\Delta t} + \frac{\mathsf{D}}{2\nu}\right) + \frac{\mathsf{D}^2\Delta t}{\nu}.$$

Using now the same arguments as for derivation of the $\ell_0^p(\mathbb{T}_N)$ estimates in Proposition 3.1, we get

$$\mathbb{E}\left[\left\|\mathbf{V}_{\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] \leq \frac{\mathsf{D}^2}{\nu^2} + \frac{\mathsf{D}^2\Delta t}{\nu} = \frac{\mathsf{D}^2}{\nu}\left(\frac{1}{\nu} + \Delta t\right).$$

To conclude, we use Inequality (51) once again:

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{V}^{N,\Delta t}\|^4_{\ell_0^4(\mathbb{T}_N)}\right] &\leq \mathbb{E}\left[\left\|\mathbf{V}_{\frac{1}{2}}^{N,\Delta t}\right\|^4_{\ell_0^4(\mathbb{T}_N)}\right] + 6\mathsf{D}\Delta t\mathbb{E}\left[\left\|\mathbf{V}_{\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right] + 3\mathsf{D}^2\Delta t^2 \\
&\leq \frac{\mathsf{D}^2}{\nu}\left(\frac{1}{\nu} + \Delta t\right) + \frac{3\mathsf{D}^2\Delta t}{\nu} + 3\mathsf{D}^2\Delta t^2 \\
&\leq \mathsf{D}^2\left(\frac{1}{\nu} + 3\Delta t_{\max}\right)\left(\frac{1}{\nu} + \Delta t_{\max}\right) =: \mathsf{C}^{\Delta,0,4}.
\end{aligned}$$

4.3. **Proof of Proposition 4.3.** Similarly to the semi-discrete scheme, we use a localisation argument in order to use the local Lipschitz continuity of the function $\mathbf{b}$. For any $M \geq 0$ and $j \geq 1$, we therefore introduce the stopping time

$$\rho^j_{(M)} = \inf\left\{t \geq 0 : \|\overline{\mathbf{U}}^{N,\Delta t_j}(t)\|_{\ell^2_0(\mathbb{T}_N)} \geq M \text{ or } \|\mathbf{U}^{N,*}(t)\|_{\ell^2_0(\mathbb{T}_N)} \geq M\right\},$$

and write, for all $t \geq 0$,

$$\begin{aligned}\mathbb{E}\left[\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\right] &= \mathbb{E}\left[\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t \leq \rho^j_{(M)}\}}\right]\\ &\quad + \mathbb{E}\left[\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t > \rho^j_{(M)}\}}\right].\end{aligned}$$

The terms in the right-hand side are respectively estimated in Lemmas 4.4 and 4.5, from which the conclusion of the proof of Proposition 4.3 is straightforward.

In the next statement, we respectively denote by $\mathsf{C}_{(M)}$ and $\mathsf{L}_{(M)}$ a bound and a Lipschitz constant (with respect to the $\ell^2_0(\mathbb{T}_N)$ norm) of $\mathbf{b}$ on the ball $\{\|\cdot\|_{\ell^2_0(\mathbb{T})} \leq M\}$.

Lemma 4.4 (Finite-time convergence in the Lipschitz case)**.** *Under the assumptions of Proposition 4.3, for all $t > 0$ and $\delta \in (0, 1]$, there exists a constant $\mathfrak{C}^\Delta(t, \mathsf{L}_{(M)}, \mathsf{C}_{(M)}, \delta, \Delta t_{\max})$ such that for any $j \geq 1$,*

$$\mathbb{E}\left[\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t \leq \rho^j_{(M)}\}}\right] \leq 2\mathrm{e}^{2\mathsf{L}_{(M)}t}\left(\mathbb{E}\left[\left\|\mathbf{U}^{N,\Delta t_j}_0 - \mathbf{U}^{N,*}_0\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right] + \Delta t_j^{1-\delta}\mathfrak{C}^\Delta(t, \mathsf{L}_{(M)}, \mathsf{C}_{(M)}, \delta, \Delta t_{\max})\right).$$

Proof. Let $t \geq 0$ and $j \geq 1$. We introduce the notation $n^j_t = \lfloor \frac{t}{\Delta t_j} \rfloor$ and first use (14) to write

$$\overline{\mathbf{U}}^{N,\Delta t_j}(t) = \mathbf{U}^{N,\Delta t_j}_{n^j_t} = \mathbf{U}^{N,\Delta t_j}_0 + \sum_{l=0}^{n^j_t - 1}\left(\mathbf{U}^{N,\Delta t_j}_{l+1} - \mathbf{U}^{N,\Delta t_j}_l\right) = \mathbf{U}^{N,\Delta t_j}_0 + \Delta t_j\sum_{l=0}^{n^j_t-1}\mathbf{b}\left(\mathbf{U}^{N,\Delta t_j}_{l+\frac12}\right) + \mathbf{W}^{Q,N}(n^j_t\Delta t_j),$$

so that, by (9),

$$\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t) = \mathbf{U}^{N,\Delta t_j}_0 - \mathbf{U}^{N,*}_0 + \sum_{l=0}^{n^j_t-1}\int_{l\Delta t_j}^{(l+1)\Delta t_j}\left(\mathbf{b}\left(\mathbf{U}^{N,\Delta t_j}_{l+\frac12}\right) - \mathbf{b}\left(\mathbf{U}^{N,*}(s)\right)\right)\mathrm{d}s - \mathbf{R}^{N,\Delta t_j}(t),$$

with

$$\mathbf{R}^{N,\Delta t_j}(t) = \int_{n^j_t\Delta t_j}^t \mathbf{b}\left(\mathbf{U}^{N,*}(s)\right)\mathrm{d}s + \mathbf{W}^{Q,N}(t) - \mathbf{W}^{Q,N}(n^j_t\Delta t_j).$$

Since, by (32), we have $\|\mathbf{U}^{N,\Delta t_j}_{l+\frac12}\|_{\ell^2_0(\mathbb{T}_N)} \leq \|\mathbf{U}^{N,\Delta t_j}_l\|_{\ell^2_0(\mathbb{T}_N)}$, we deduce that if $t \leq \rho^j_{(M)}$, then for any $l \leq n^j_t - 1$ and $s \in [l\Delta t_j, (l+1)\Delta t_j]$,

$$\begin{aligned}\left\|\mathbf{b}\left(\mathbf{U}^{N,\Delta t_j}_{l+\frac12}\right) - \mathbf{b}\left(\mathbf{U}^{N,*}(s)\right)\right\|_{\ell^2_0(\mathbb{T}_N)} &\leq \mathsf{L}_{(M)}\left\|\mathbf{U}^{N,\Delta t_j}_{l+\frac12} - \mathbf{U}^{N,*}(s)\right\|_{\ell^2_0(\mathbb{T}_N)}\\ &\leq \mathsf{L}_{(M)}\left(\left\|\mathbf{U}^{N,\Delta t_j}_{l+\frac12} - \mathbf{U}^{N,\Delta t_j}_l\right\|_{\ell^2_0(\mathbb{T}_N)} + \left\|\overline{\mathbf{U}}^{N,\Delta t_j}(s) - \mathbf{U}^{N,*}(s)\right\|_{\ell^2_0(\mathbb{T}_N)}\right)\\ &\leq \mathsf{L}_{(M)}\left(\Delta t_j\mathsf{C}_{(M)} + \left\|\overline{\mathbf{U}}^{N,\Delta t_j}(s) - \mathbf{U}^{N,*}(s)\right\|_{\ell^2_0(\mathbb{T}_N)}\right).\end{aligned}$$

Likewise, rewriting

$$\mathbf{R}^{N,\Delta t_j}(t) = \int_{n^j_t\Delta t_j}^t\left(\mathbf{b}\left(\mathbf{U}^{N,*}(s)\right) - \mathbf{b}\left(\overline{\mathbf{U}}^{N,\Delta t_j}(s)\right)\right)\mathrm{d}s + (t - n_t\Delta t_j)\,\mathbf{b}\left(\mathbf{U}^{N,\Delta t_j}_{n^j_t}\right) + \mathbf{W}^{Q,N}(t) - \mathbf{W}^{Q,N}(n^j_t\Delta t_j)$$

we get

$$\left\|\mathbf{R}^{N,\Delta t_j}(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \leq \mathsf{L}_{(M)}\int_{n^j_t\Delta t_j}^t\left\|\overline{\mathbf{U}}^{N,\Delta t_j}(s) - \mathbf{U}^{N,*}(s)\right\|_{\ell^2_0(\mathbb{T}_N)}\mathrm{d}s + \Delta t_j\mathsf{C}_{(M)} + \left\|\mathbf{W}^{Q,N}(t) - \mathbf{W}^{Q,N}(n^j_t\Delta t_j)\right\|_{\ell^2_0(\mathbb{T}_N)}.$$

We deduce that

$$\left\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \leq \left\|\mathbf{U}^{N,\Delta t_j}_0 - \mathbf{U}^{N,*}_0\right\|_{\ell^2_0(\mathbb{T}_N)} + \mathsf{L}_{(M)}\int_0^t\left\|\overline{\mathbf{U}}^{N,\Delta t_j}(s) - \mathbf{U}^{N,*}(s)\right\|_{\ell^2_0(\mathbb{T}_N)}\mathrm{d}s + \mathfrak{c}^{\Delta,j}(t),$$

with

$$\mathfrak{c}^{\Delta,j}(t) = \Delta t_j\mathsf{C}_{(M)}\left(\mathsf{L}_{(M)}t + 1\right) + \max_{l=0,\ldots,n^j_t}\sup_{s\in[l\Delta t_j,(l+1)\Delta t_j)}\left\|\mathbf{W}^{Q,N}(s) - \mathbf{W}^{Q,N}(l\Delta t_j)\right\|_{\ell^2_0(\mathbb{T}_N)}.$$

Therefore, by Grönwall's lemma,

$$\left\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \leq \mathrm{e}^{\mathsf{L}_{(M)}t}\left(\left\|\mathbf{U}^{N,\Delta t_j}_0 - \mathbf{U}^{N,*}_0\right\|_{\ell^2_0(\mathbb{T}_N)} + \mathfrak{c}^{\Delta,j}(t)\right),$$

and then by Jensen's inequality,

$$\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} \mathbb{1}_{\{t \leq \rho^j_{(M)}\}}\right] \leq 2\mathrm{e}^{2\mathsf{L}_{(M)}t}\left(\mathbb{E}\left[\left\|\mathbf{U}^{N,\Delta t_j}_0 - \mathbf{U}^{N,*}_0\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right] + \mathbb{E}\left[\mathfrak{c}^{\Delta,j}(t)^2\right]\right).$$

It remains to estimate

$$\mathbb{E}\left[\mathfrak{c}^{\Delta,j}(t)^2\right] \leq 2\left(\Delta t_j^2 \mathsf{C}^2_{(M)}\left(\mathsf{L}_{(M)}t+1\right)^2 + \mathbb{E}\left[\max_{l=0,\dots,n^j_t}\sup_{s\in[l\Delta t_j,(l+1)\Delta t_j)}\left\|\mathbf{W}^{Q,N}(s) - \mathbf{W}^{Q,N}(l\Delta t_j)\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\right).$$

By the Markov property and scaling invariance for the Wiener process $\mathbf{W}^{Q,N}$, the random variables

$$G_l := \sup_{s\in[l\Delta t_j,(l+1)\Delta t_j)}\left\|\mathbf{W}^{Q,N}(s) - \mathbf{W}^{Q,N}(l\Delta t_j)\right\|^2_{\ell^2_0(\mathbb{T}_N)}, \qquad l \geq 0,$$

are independent and identically distributed, with, for any $p \in [1,+\infty)$,

$$\mathbb{E}[G_l^p] = \Delta t_j^p \mathsf{g}_p,$$

for some constant g_p which depends on p and the vectors $\mathbf{g}^k$ but not on Δt_j. As a consequence, we deduce from Jensen's inequality that

$$\mathbb{E}\left[\max_{l=0,\dots,n^j_t} G_l\right] \leq \mathbb{E}\left[\max_{l=0,\dots,n^j_t} G_l^p\right]^{1/p} \leq \mathbb{E}\left[\sum_{l=0}^{n^j_t} G_l^p\right]^{1/p} = (n^j_t+1)^{1/p}\Delta t_j \mathsf{g}_p^{1/p} \leq (t+\Delta t_{\max})\Delta t_j^{1-1/p}\mathsf{g}_p^{1/p},$$

which yields the claimed estimate by taking $p = 1/\delta$. □

Lemma 4.5 (Uniform control over $\rho^j_{(M)}$). *Under the assumptions of Proposition 4.3, for any $t \geq 0$,*

$$\lim_{M\to+\infty}\limsup_{j\to+\infty}\mathbb{E}\left[\|\overline{\mathbf{U}}^{N,\Delta t_j}(t) - \mathbf{U}^{N,*}(t)\|^2_{\ell^2_0(\mathbb{T}_N)}\mathbb{1}_{\{t>\rho^j_{(M)}\}}\right] = 0.$$

The proof of Lemma 4.5 follows the same outline as Lemma 3.8. The $\ell^4_0(\mathbb{T}_N)$ bounds over $\overline{\mathbf{U}}^{N,\Delta t_j}(t)$ and $\mathbf{U}^{N,*}(t)$ respectively follow from Proposition 4.1 and (28), in which $\mathbb{E}[\|\mathbf{U}^{N,*}_0\|^4_{\ell^4_0(\mathbb{T}_N)}]$ is bounded from above by $\mathsf{C}^{\Delta,0,4}$ thanks to Corollary 4.2. The uniform (in j) bound on $\mathbb{E}[\sup_{s\in[0,t]}\|\overline{\mathbf{U}}^{N,\Delta t_j}(t)\|^2_{\ell^2_0(\mathbb{T}_N)}]$ is stated in the next lemma, the proof of which is postponed to Appendix C.

Lemma 4.6 (Finite-time uniform $\ell^2_0(\mathbb{T}_N)$ bound on $\overline{\mathbf{U}}^{N,\Delta t_j}$). *Under the assumptions of Proposition 4.3, for all $t \geq 0$ there exists a constant $\mathsf{S}^{\Delta,0,2}_t$, which does not depend on Δt_j, such that*

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left\|\overline{\mathbf{U}}^{N,\Delta t_j}(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right] \leq \mathsf{S}^{\Delta,0,2}_t.$$

In order to complete the proof of Proposition 4.3, it only remains to check that

$$\lim_{j\to+\infty}\mathbb{E}\left[\left\|\mathbf{U}^{N,\Delta t_j}_0 - \mathbf{U}^{N,*}_0\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right] = 0,$$

which follows from the $\ell^4_0(\mathbb{T}_N)$ estimates from Proposition 3.1 and Corollary 4.2 by uniform integrability.

4.4. **Discussion of the rate of convergence.** Unlike Lemma 3.7, our proof of Lemma 4.4 is not designed to yield a strong error estimate valid in the long time limit, because it does not directly exploit the decomposition of $\mathbf{b}$ into dissipative and contractive parts. Let us however point out the fact that all constants involved in the uniform in Δt estimates in this section also turn out to be uniform in N, which is not crucial for our purpose since we establish results for a fixed value of N, but might indicate that both limits $\Delta t \to 0$ and $N \to +\infty$, and associated rates of convergence, could be studied simultaneously.

As far as weak error estimates are concerned, one might expect in the Lipschitz case a *weak* error of order Δt between $\nu_{N,\Delta t}$ and ν_N, as is the case for explicit Euler schemes [30]. For gradient SDEs with a non globally Lipschitz continuous drift, the weak backward error analysis of split-step schemes also shows order Δt [25]. For the SDE (9), we also observe order Δt on the numerical simulations of Section 5, even with a non small (with respect to ν) and non Lipschitz continuous flux function A.

5. Numerical experiments

5.1. **Analytic case.** In this subsection, we consider the case where the flux function A vanishes, so that (1) is the stochastic heat equation

$$\mathrm{d}u(t) = \nu\partial_{xx}u(t)\mathrm{d}t + \sum_{k\geq 1} g^k \mathrm{d}W^k(t), \tag{54}$$

and the SDE (9) writes

$$\mathrm{d}\mathbf{U}^N(t) = \nu\mathbf{D}_N^{(2)}\mathbf{U}^N(t)\mathrm{d}t + \sum_{k\geq 1}\mathbf{g}^k\mathrm{d}W^k(t). \tag{55}$$

5.1.1. *Computation of μ and ν_N.* Since the drift of the SDE (55) is linear, the process $(\mathbf{U}^N(t))_{t\geq 0}$ is Gaussian, and its stationary distribution ν_N is the centered Gaussian measure on $\mathbb{R}_0^N$ with covariance matrix $\mathbf{K}_N$ solution to the Lyapunov equation

$$\nu\mathbf{K}_N\mathbf{D}_N^{(2)} + (\nu\mathbf{K}_N\mathbf{D}_N^{(2)})^\top + \mathbf{Q}_N = 0, \qquad \mathbf{Q}_N := \sum_{k\geq 1}\mathbf{g}^k{\mathbf{g}^k}^\top.$$

The solution to this equation admits the explicit expression

$$\mathbf{K}_N = \int_0^{+\infty} \mathrm{e}^{t\nu\mathbf{D}_N^{(2)}}\mathbf{Q}_N\mathrm{e}^{t\nu\mathbf{D}_N^{(2)}}\mathrm{d}t = \sum_{k\geq 1}\int_0^{+\infty}\boldsymbol{\rho}_N^k(t)\boldsymbol{\rho}_N^k(t)^\top\mathrm{d}t,$$

where $\boldsymbol{\rho}_N^k(t) = \mathrm{e}^{t\nu\mathbf{D}_N^{(2)}}\mathbf{g}^k$ is the solution to the discrete heat equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho}_N^k(t) = \nu\mathbf{D}_N^{(2)}\boldsymbol{\rho}_N^k(t), \qquad \boldsymbol{\rho}_N^k(0) = \mathbf{g}^k.$$

Using the duality relation

$$\langle v, \Psi_N\mathbf{w}\rangle_{L_0^2(\mathbb{T}_N)} = \langle \Pi_N v, \mathbf{w}\rangle_{\ell_0^2(\mathbb{T}_N)}, \qquad v\in L_0^2(\mathbb{T}), \quad \mathbf{w}\in\mathbb{R}_0^N,$$

we deduce that the pushforward measure μ_N is the centered Gaussian measure on $L_0^2(\mathbb{T})$ with covariance operator $K_N : L_0^2(\mathbb{T}) \to L_0^2(\mathbb{T})$ defined by, for any $v, w \in L_0^2(\mathbb{T})$,

$$\begin{aligned}\langle v, K_N w\rangle_{L_0^2(\mathbb{T})} &= \sum_{k\geq 1}\int_0^{+\infty}\langle\boldsymbol{\rho}_N^k(t), \Pi_N v\rangle_{\ell_0^2(\mathbb{T}_N)}\langle\boldsymbol{\rho}_N^k(t), \Pi_N w\rangle_{\ell_0^2(\mathbb{T}_N)}\mathrm{d}t\\ &= \sum_{k\geq 1}\int_0^{+\infty}\langle\Psi_N\boldsymbol{\rho}_N^k(t), v\rangle_{L_0^2(\mathbb{T})}\langle\Psi_N\boldsymbol{\rho}_N^k(t), w\rangle_{L_0^2(\mathbb{T})}\mathrm{d}t.\end{aligned}$$

A similar computation at the infinite-dimensional level of (54) shows that μ is the centered Gaussian measure on $L_0^2(\mathbb{T})$ with covariance operator $K : L_0^2(\mathbb{T}) \to L_0^2(\mathbb{T})$ defined by, for any $v, w \in L_0^2(\mathbb{T})$,

$$\langle v, Kw\rangle_{L_0^2(\mathbb{T})} = \sum_{k\geq 1}\int_0^{+\infty}\langle r^k(t), v\rangle_{L_0^2(\mathbb{T})}\langle r^k(t), w\rangle_{L_0^2(\mathbb{T})}\mathrm{d}t,$$

where $r^k(t)$ is the solution to the heat equation

$$\partial_t r^k(t,x) = \nu\partial_{xx}r^k(t,x), \qquad r^k(0,x) = g^k(x).$$

5.1.2. *Computation of $W_2(\mu_N, \mu)$.* In order to compute explicitly $W_2(\mu_N, \mu)$, we now assume that $g^k = 0$ for $k \geq 2$, and take

$$g^1(x) = g(x) := \sqrt{2}\sin\left(2\pi m_0 x\right),$$

for some $m_0 \in \mathbb{N}^*$. The main advantage of this choice lies in the spectral identities

$$\partial_{xx}g = -\lambda g, \qquad \lambda = (2\pi m_0)^2,$$

and, with $\mathbf{g} = \Pi_N g$,

$$\mathbf{D}_N^{(2)}\mathbf{g} = -\lambda_N\mathbf{g}, \qquad \lambda_N = 2N^2\left(1-\cos\left(\frac{2\pi m_0}{N}\right)\right).$$

These identities allow to compute the time integrals appearing in the operators K and K_N and yield, for any $v, w \in L_0^2(\mathbb{T})$,

$$\langle v, Kw\rangle_N = \frac{1}{2\nu\lambda}\langle g, v\rangle_{L_0^2(\mathbb{T})}\langle g, w\rangle_{L_0^2(\mathbb{T})}, \qquad \langle v, K_N w\rangle_N = \frac{1}{2\nu\lambda_N}\langle\Psi_N\mathbf{g}, v\rangle_{L_0^2(\mathbb{T})}\langle\Psi_N\mathbf{g}, w\rangle_{L_0^2(\mathbb{T})}.$$

As a consequence, both operators K and K_N have rank 1, and it follows from standard results on finite-dimensional Gaussian vectors [19] that the optimal coupling between μ and μ_N in Definition 1.6 is given by the law of the pair of random variables (u, u^N) defined by

$$u = \frac{Z}{\sqrt{2\nu\lambda}} g, \qquad u^N = \varepsilon_N \frac{Z}{\sqrt{2\nu\lambda_N}} \Psi_N \mathbf{g}, \qquad \varepsilon_N := \operatorname{sign}\left(\langle g, \Psi_N \mathbf{g}\rangle_{L^2_0(\mathbb{T})}\right),$$

where Z is a standard, one-dimensional Gaussian variable. The Wasserstein distance $W_2(\mu_N, \mu)$ then writes

$$W_2(\mu_N, \mu) = \left\| \frac{g}{\sqrt{2\nu\lambda}} - \frac{\varepsilon_N \Psi_N \mathbf{g}}{\sqrt{2\nu\lambda_N}} \right\|_{L^2_0(\mathbb{T})}.$$

For N large enough, $\varepsilon_N = 1$ and $\lambda_N = \lambda + \mathrm{O}(1/N^2)$, so that

$$N W_2(\mu_N, \mu) \sim \frac{N}{\sqrt{2\nu\lambda}} \|g - \Psi_N \mathbf{g}\|_{L^2_0(\mathbb{T})} \to \frac{1}{\sqrt{24\nu\lambda}} \|g\|_{H^1_0(\mathbb{T})},$$

which confirms that the rate $1/N$ derived in Subsection 3.4 is sharp in this case.

5.1.3. *Time discretisation.* The split-step scheme associated with (55) rewrites

$$\mathbf{U}^{N,\Delta t}_{n+1} = \left(\mathbf{I} - \nu\Delta t \mathbf{D}^{(2)}_N\right)^{-1} \mathbf{U}^{N,\Delta t}_n + \Delta \mathbf{W}^{Q,N}_{n+1},$$

which shows that its invariant measure is the centered Gaussian measure on $\mathbb{R}^N_0$ with covariance matrix $\mathbf{K}_{N,\Delta t}$ given by

$$\mathbf{K}_{N,\Delta t} = \Delta t \sum_{n=0}^{+\infty} \left(\left(\mathbf{I} - \nu\Delta t \mathbf{D}^{(2)}_N\right)^{-1}\right)^n \mathbf{Q}_N \left(\left(\mathbf{I} - \nu\Delta t \mathbf{D}^{(2)}_N\right)^{-1}\right)^n.$$

With the one-dimensional noise introduced above, this expression reduces to

$$\mathbf{K}_{N,\Delta t} = \Delta t \frac{(1 + \nu\Delta t\lambda_N)^2}{(1 + \nu\Delta t\lambda_N)^2 - 1} \mathbf{g}\mathbf{g}^\top.$$

It follows that

$$W_2(\nu_{N,\Delta t}, \nu_N) = \left| \sqrt{\frac{1}{2\nu\lambda_N}} - \sqrt{\Delta t \frac{(1 + \nu\Delta t\lambda_N)^2}{(1 + \nu\Delta t\lambda_N)^2 - 1}} \right| \|\mathbf{g}\|_{\ell^2_0(\mathbb{T}_N)} = \Delta t (1 + \mathrm{O}(\Delta t)) \sqrt{\frac{\nu\lambda_N}{2}} \|\mathbf{g}\|_{\ell^2_0(\mathbb{T}_N)},$$

which shows the order Δt for the Wasserstein distance between $\nu_{N,\Delta t}$ and ν_N, uniformly in N.

5.2. **Numerical experiments.** In this subsection, we study numerically the *weak error* between $\nu_{N,\Delta t}$ and ν_N, as a function of Δt. More precisely, we take as a test function

$$\Phi(\mathbf{v}) = \exp\left(-\|\mathbf{v}\|^2_{\ell^2_0(\mathbb{T}_N)}\right), \quad \mathbf{v} \in \mathbb{R}^N_0,$$

and estimate

$$\mathtt{err}_N(\Delta t) := \left|\mathbb{E}\left[\Phi(\mathbf{V}^{N,\Delta t})\right] - \mathbb{E}\left[\Phi(\mathbf{V}^N)\right]\right|, \qquad \mathbf{V}^{N,\Delta t} \sim \nu_{N,\Delta t}, \quad \mathbf{V}^N \sim \nu_N.$$

Notice that, since Φ is globally Lipschitz continuous, the quantity $\mathtt{err}_N(\Delta t)$ is controlled by $W_2(\nu_{N,\Delta t}, \nu_N)$ and is therefore at most of order Δt in the case $\alpha = 0$.

We work with the Burgers equation, for which $A(v) = \alpha v^2/2$, with a strength parameter $\alpha \geq 0$. We keep the one-dimensional noise introduced in Subsection 5.1 and take the values $m_0 = 1$ in order to minimise spatial oscillations, and $\nu = 0.1$.

In the case $\alpha = 0$, under the invariant measure μ, the functions $\partial_{xx} u$ and $u \partial_x u$ have respective orders of magnitude $\nu^{-1/2}$ and ν^{-1} in the $L^2_0(\mathbb{T})$ norm. Therefore, we shall study:

- the linear regime $\alpha = 0$;
- the viscous regime $|\alpha u \partial_x u| \ll |\nu \partial_{xx} u|$, that is to say $\alpha \ll \nu^{3/2}$;
- the equilibrated regime $|\alpha u \partial_x u| \simeq |\nu \partial_{xx} u|$, that is to say $\alpha \simeq \nu^{3/2}$;
- the inviscid regime $|\alpha u \partial_x u| \gg |\nu \partial_{xx} u|$, that is to say $\alpha \gg \nu^{3/2}$.

5.2.1. *Analytic results for $\alpha = 0$.* In the Gaussian case $\alpha = 0$, the expectations involved in the definition of $\mathtt{err}_N(\Delta t)$ are analytic and write, with the notation of Subsection 5.1,

$$\begin{aligned}
\mathbb{E}[\Phi(\mathbf{V}^{N,\Delta t})] &= \sqrt{\frac{1}{1+2\kappa_{N,\Delta t}}}, \qquad \kappa_{N,\Delta t} = \frac{\Delta t(1+\nu\Delta t\lambda_N)^2}{(1+\nu\Delta t\lambda_N)^2-1}\|\mathbf{g}\|^2_{\ell^2_0(\mathbb{T}_N)},\\
\mathbb{E}[\Phi(\mathbf{V}^{N})] &= \sqrt{\frac{1}{1+2\kappa_{N}}}, \qquad \kappa_{N} := \frac{\|\mathbf{g}\|^2_{\ell^2_0(\mathbb{T}_N)}}{2\nu\lambda_N},
\end{aligned}$$

with $\|\mathbf{g}\|^2_{\ell^2_0(\mathbb{T}_N)} = (\sin(\frac{\pi m_0}{N})/\frac{\pi m_0}{N})^2$ as soon as $N > 2m_0$. We deduce from these expressions that $\mathtt{err}_N(\Delta t)$ is of order Δt, uniformly in N.

5.2.2. *Ergodic approximation of $\mathbb{E}[\Phi(\mathbf{V}^{N,\Delta t})]$.* For $N \geq 1$ and $\Delta t > 0$, let us denote

$$I^{N,\Delta t} := \mathbb{E}[\Phi(\mathbf{V}^{N,\Delta t})].$$

From Remark 1.9 and the Central Limit Theorem for Markov chains, we may expect that there exist $\Sigma^{N,\Delta t}$ such that for T large enough, the random variable

$$\widehat{I}^{N,\Delta t}_T = \frac{1}{n}\sum_{l=0}^{n-1}\Phi(\mathbf{U}^{N,\Delta t}_l), \qquad T = n\Delta t,$$

has the Gaussian distribution

$$\widehat{I}^{N,\Delta t}_T \sim \mathcal{N}\left(I^{N,\Delta t}, \frac{(\Sigma^{N,\Delta t})^2}{T}\right).$$

The evolution of the empirical average $\widehat{I}^{N,\Delta t}_t$, $t \in [0,T]$, along a single trajectory of the split-step scheme is plotted on Figure 1 for the four regimes of α. The value of $I^{N,\Delta t}$ seems to be the same for all regimes $\alpha = 0$, $\alpha \ll \nu^{3/2}$ and $\alpha \simeq \nu^{3/2}$, and to be significantly larger for $\alpha \gg \nu^{3/2}$.

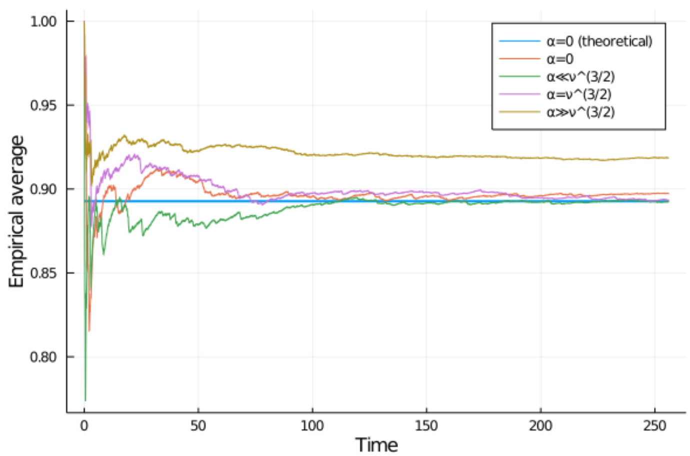

FIGURE 1. Evolution of $\widehat{I}^{N,\Delta t}_t$ for $t \in [0,T]$ and the four regimes of α. In the case $\alpha = 0$, the theoretical value of $I^{N,\Delta t}$ is superposed as a horizontal line. Here, $N = 32$, $\Delta t = 2^{-10}$ and $T = 256$.

In § 5.2.3, Monte Carlo confidence intervals for $I^{N,\Delta t}$ are obtained by fixing a time horizon $T \gg 1$ and estimating the parameters $I^{N,\Delta t}$ and $\Sigma^{N,\Delta t}$ from a sample of $M \gg 1$ independent realisations $\widehat{I}^{N,\Delta t,(1)}_T, \ldots, \widehat{I}^{N,\Delta t,(M)}_T$.

5.2.3. *Weak error between $\nu_{N,\Delta t}$ and ν_N.* Let $N = 32$, $\Delta t_{\min} = 2^{-8}$ and $\Delta t_{\max} = 2^{-1}$. Our purpose is to plot the evolution of $\mathtt{err}_N(\Delta t)$ for $\Delta t \in [\Delta t_{\min}, \Delta t_{\max}]$. To this aim, we fix $\overline{\Delta t} = 2^{-10} \ll \Delta t_{\min}$, approximate

$$\mathtt{err}_N(\Delta t) \simeq \left|I^{N,\Delta t} - I^{N,\overline{\Delta t}}\right|,$$

and compute both terms in the right-hand side thanks to Monte Carlo simulations as is described in § 5.2.2.

The resulting error curves are plotted on Figure 2 for the four regimes of α. The weak error is observed to be of order Δt, uniformly in α.

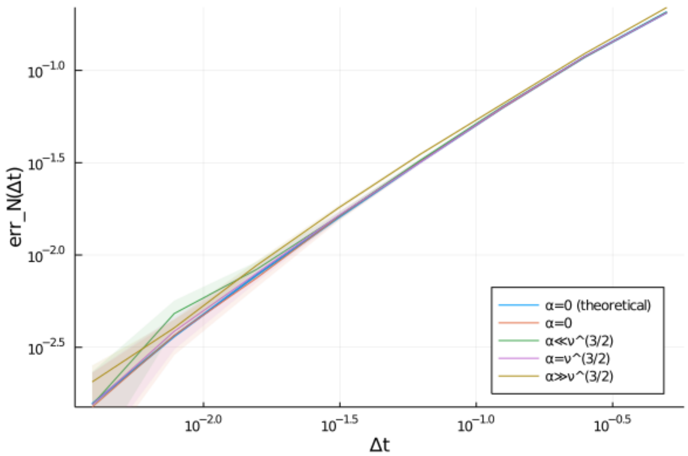

FIGURE 2. Evolution of $\mathtt{err}_N(\Delta t)$ for $\Delta t \in [\Delta t_{\min}, \Delta t_{\max}]$ and the four regimes of α, with associated confidence intervals. In the case $\alpha = 0$, the theoretical value of $\mathtt{err}_N(\Delta t)$ is superposed. Here, the final time horizon is $T = 256$ and the number of copies for the Monte Carlo estimation is $M = 200$.

Remark 5.1. The error curves of Figure 2 are plotted with the same final time horizon T, and the same number of Monte Carlo realisations M, for all values of Δt. As it turns out that the asymptotic variance $\Sigma^{N,\Delta t}$ is approximately uniform in N and Δt, this results in the estimator of $\mathtt{err}_N(\Delta t)$ having the same variance for all Δt. This is the reason why, in log-log coordinates, confidence intervals appear to be larger for smaller values of Δt.

Appendix A. Proofs of auxiliary inequalities

Proof of Lemma 2.2. Let $\mathbf{v} \in \mathbb{R}_0^N$ and $q \in 2\mathbb{N}^*$. By (19) we have

$$\langle \mathbf{v}^{q-1}, \mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{v})\rangle_{\ell^2(\mathbb{T}_N)} = -\langle \mathbf{D}^{(1,+)}\mathbf{v}^{q-1}, \overline{\mathbf{A}}^N(\mathbf{v})\rangle_{\ell^2(\mathbb{T}_N)} = -\sum_{i\in\mathbb{T}_N}(v_{i+1}^{q-1} - v_i^{q-1})\overline{A}(v_i, v_{i+1}).$$

For any $i \in \mathbb{T}_N$, using (11) and (10), we get

$$(v_{i+1}^{q-1} - v_i^{q-1})\overline{A}(v_i, v_{i+1}) = \int_{v_i^{q-1}}^{v_{i+1}^{q-1}} \overline{A}(v_i, v_{i+1})\mathrm{d}z \le \int_{v_i^{q-1}}^{v_{i+1}^{q-1}} \overline{A}\left(z^{1/(q-1)}, z^{1/(q-1)}\right)\mathrm{d}z = \mathcal{A}_q\left(v_{i+1}^{q-1}\right) - \mathcal{A}_q\left(v_i^{q-1}\right),$$

where $\mathcal{A}_q$ denotes a function defined on $\mathbb{R}$ such that $\mathcal{A}_q'(z) = A(z^{1/(q-1)})$. Since the sum over $i \in \mathbb{T}_N$ of all terms in the right-hand side vanish, the proof is completed. $\square$

Proof of Lemma 2.6. For $\mathbf{v} \in \mathbb{R}_0^N$ and $p \in 2\mathbb{N}^*$, we have by Jensen's inequality

$$\begin{aligned}
\left\langle \mathbf{D}_N^{(1,+)}(\mathbf{v}^{p-1}), \mathbf{D}_N^{(1,+)}\mathbf{v}\right\rangle_{\ell_0^2(\mathbb{T}_N)} &= N\sum_{i\in\mathbb{T}_N}\left(v_{i+1}^{p-1} - v_i^{p-1}\right)(v_{i+1} - v_i)\\
&= N(p-1)\sum_{i\in\mathbb{T}_N}(v_{i+1} - v_i)\int_{v_i}^{v_{i+1}}|z|^{p-2}\mathrm{d}z\\
&= N(p-1)\sum_{i\in\mathbb{T}_N}(v_{i+1} - v_i)\int_{v_i}^{v_{i+1}}\left(|z|^{p/2-1}\right)^2\mathrm{d}z\\
&\ge N(p-1)\sum_{i\in\mathbb{T}_N}\left(\int_{v_i}^{v_{i+1}}|z|^{p/2-1}\mathrm{d}z\right)^2\\
&= \frac{4N(p-1)}{p^2}\sum_{i\in\mathbb{T}_N}\left(\int_{v_i}^{v_{i+1}}\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{sign}(z)|z|^{p/2}\right)\mathrm{d}z\right)^2\\
&= \frac{4N(p-1)}{p^2}\sum_{i\in\mathbb{T}_N}\left(\mathrm{sign}\,(v_{i+1})\,|v_{i+1}|^{p/2} - \mathrm{sign}\,(v_i)\,|v_i|^{p/2}\right)^2\\
&= \frac{4(p-1)}{p^2}\|\mathbf{D}^{(1,+)}\mathbf{w}\|^2_{\ell_0^2(\mathbb{T}_N)},
\end{aligned}$$

where $\mathbf{w}$ is the vector with coordinates $w_i = \mathrm{sign}\,(v_i)\,|v_i|^{p/2}$, $i \in \mathbb{T}_N$. This vector does not necessarily belong to $\mathbb{R}^N_0$ so we cannot apply the Poincaré inequality (22) direclty. Let us however notice that since $\mathbf{v} \in \mathbb{R}^N_0$, there exist two indices $i_-, i_+ \in \mathbb{T}_N$ such that $v_{i_-} \leq 0 \leq v_{i_+}$, so that $w_{i_-} \leq 0 \leq w_{i_+}$. As a consequence,

$$\begin{aligned}
\|\mathbf{w}\|^2_{\ell^2_0(\mathbb{T}_N)} &= \frac{1}{N}\sum_{w_i \geq 0} |w_i|^2 + \frac{1}{N}\sum_{w_i<0} |w_i|^2 \\
&\leq \frac{1}{N}\sum_{w_i \geq 0} |w_i - w_{i_-}|^2 + \frac{1}{N}\sum_{w_i<0} |w_i - w_{i_+}|^2 \\
&\leq \frac{1}{N}\sum_{w_i \geq 0} N \sum_{j\in\mathbb{T}_N} |w_{j+1} - w_j|^2 + \frac{1}{N}\sum_{w_i<0} N \sum_{j\in\mathbb{T}_N} |w_{j+1} - w_j|^2 \\
&= N \sum_{j\in\mathbb{T}_N} |w_{j+1} - w_j|^2 \\
&= \left\| \mathbf{D}^{(1,+)}_N \mathbf{w} \right\|^2_{\ell^2_0(\mathbb{T}_N)}.
\end{aligned}$$

Injecting this estimate in the identity above, we deduce that

$$\left\langle \mathbf{D}^{(1,+)}_N(\mathbf{v}^{p-1}), \mathbf{D}^{(1,+)}_N \mathbf{v} \right\rangle_{\ell^2_0(\mathbb{T}_N)} \geq \frac{4(p-1)}{p^2} \|\mathbf{w}\|^2_{\ell^2_0(\mathbb{T}_N)} = \frac{4(p-1)}{p^2} \|\mathbf{v}\|^p_{\ell^p_0(\mathbb{T}_N)},$$

which completes the proof. □

Appendix B. Proofs of intermediary results for the uniqueness of invariant measures

B.1. The semi-discrete scheme.

Proof of Lemma 2.10. We recall that $\mathbf{b} : \mathbb{R}^N_0 \to \mathbb{R}^N_0$ is locally Lipschitz continuous (for every norm over $\mathbb{R}^N_0$). Let $M > 0$ and $\varepsilon > 0$. Let us also fix the deterministic values $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}^N_0$ satisfying $\|\mathbf{u}_0\|_{\ell^2_0(\mathbb{T}_N)} \vee \|\mathbf{v}_0\|_{\ell^2_0(\mathbb{T}_N)} \leq M$, along with the following constants:

$$t_{\varepsilon,M} := -\frac{1}{2\nu} \log \frac{\varepsilon^2}{16M^2};$$

$$\mathsf{L}_{M+\varepsilon} := \text{ Lipschitz constant of } \mathbf{b} \text{ over the ball } \left\{ \|\cdot\|_{\ell^1_0(\mathbb{T}_N)} \leq M + \varepsilon \right\};$$

$$\delta_\varepsilon := \frac{\varepsilon}{4} \mathrm{e}^{-\mathsf{L}_{M+\varepsilon} t_{\varepsilon,M}}.$$

Let $(\mathbf{U}^N(t))_{t\geq 0}$ and $(\mathbf{V}^N(t))_{t\geq 0}$ denote two solutions of (9) with the initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$. We introduce the stopping times

$$\tilde{\tau}^{\mathbf{U}^N} := \inf\left\{ t \geq 0 : \|\mathbf{U}^N(t)\|_{\ell^1_0(\mathbb{T}_N)} \geq M + \varepsilon \right\}, \qquad \tilde{\tau}^{\mathbf{V}^N} := \inf\left\{ t \geq 0 : \|\mathbf{V}^N(t)\|_{\ell^1_0(\mathbb{T}_N)} \geq M + \varepsilon \right\}.$$

Furthermore, we denote by $(\mathbf{u}^N(t))_{t\geq 0}$ and $(\mathbf{v}^N(t))_{t\geq 0}$ the noiseless counterparts of $(\mathbf{U}^N(t))_{t\geq 0}$ and $(\mathbf{V}^N(t))_{t\geq 0}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^N(t) = \mathbf{b}\left(\mathbf{u}^N(t)\right), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^N(t) = \mathbf{b}\left(\mathbf{v}^N(t)\right),$$

with respective initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$.

By Lemma 2.1.(ii) and (22), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\mathbf{u}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} + \left\|\mathbf{v}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} \right) \leq -2\nu \left(\left\|\mathbf{u}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} + \left\|\mathbf{v}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} \right),$$

so that Grönwall's lemma yields the upper bound

$$\left\|\mathbf{u}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} + \left\|\mathbf{v}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} \leq \left(\|\mathbf{u}_0\|^2_{\ell^2_0(\mathbb{T}_N)} + \|\mathbf{v}_0\|^2_{\ell^2_0(\mathbb{T}_N)} \right) \mathrm{e}^{-2\nu t},$$

meaning that for $t \geq t_{\varepsilon,M}$, we have

$$\left\|\mathbf{u}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} + \left\|\mathbf{v}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} \leq \frac{\varepsilon^2}{8},$$

and consequently, by (21),

$$\left\|\mathbf{u}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{v}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \leq \left\|\mathbf{u}^N(t)\right\|_{\ell^2_0(\mathbb{T}_N)} + \left\|\mathbf{v}^N(t)\right\|_{\ell^2_0(\mathbb{T}_N)} \leq \frac{\varepsilon}{2}.$$

We now restrict ourselves to the event

$$\left\{ \sup_{t\in[0,t_{\varepsilon,M}]} \left\|\mathbf{W}^{Q,N}(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \leq \delta_\varepsilon \right\}.$$

For any $t \le \widetilde{\tau}^{\mathbf{U}^N} \wedge \widetilde{\tau}^{\mathbf{V}^N} \wedge t_{\varepsilon,M}$, the four vectors $\mathbf{U}^N(t)$, $\mathbf{V}^N(t)$, $\mathbf{u}^N(t)$ and $\mathbf{v}^N(t)$ stay in the ball $\{\|\cdot\|_{\ell^1_0(\mathbb{T}_N)} \le M+\varepsilon\}$, and thanks to the local Lipschitz continuity assumption on $\mathbf{b}$ we have

$$
\begin{aligned}
&\left\|\mathbf{U}^N(t) - \mathbf{u}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(t) - \mathbf{v}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \\
&= \left\|\int_0^t \left(\mathbf{b}\left(\mathbf{U}^N(s)\right) - \mathbf{b}\left(\mathbf{u}^N(s)\right)\right) \mathrm{d}s + \mathbf{W}^{Q,N}(t)\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\int_0^t \left(\mathbf{b}\left(\mathbf{V}^N(s)\right) - \mathbf{b}\left(\mathbf{v}^N(s)\right)\right) \mathrm{d}s + \mathbf{W}^{Q,N}(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \\
&\le \int_0^t \left(\left\|\mathbf{b}\left(\mathbf{U}^N(s)\right) - \mathbf{b}\left(\mathbf{u}^N(s)\right)\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{b}\left(\mathbf{V}^N(s)\right) - \mathbf{b}\left(\mathbf{v}^N(s)\right)\right\|_{\ell^1_0(\mathbb{T}_N)}\right) \mathrm{d}s + 2\left\|\mathbf{W}^{Q,N}(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \\
&\le \mathsf{L}_{M+\varepsilon} \int_0^t \left(\left\|\mathbf{U}^N(s) - \mathbf{u}^N(s)\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(s) - \mathbf{v}^N(s)\right\|_{\ell^1_0(\mathbb{T}_N)}\right) \mathrm{d}s + 2\delta_\varepsilon,
\end{aligned}
$$

so by Grönwall's lemma, we have

$$
\left\|\mathbf{U}^N(t) - \mathbf{u}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(t) - \mathbf{v}^N(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \le 2\delta_\varepsilon \mathrm{e}^{\mathsf{L}_{M+\varepsilon} t} \le 2\delta_\varepsilon \mathrm{e}^{\mathsf{L}_{M+\varepsilon} t_{\varepsilon,M}} = \frac{\varepsilon}{2}, \tag{56}
$$

for every $t \in [0, \widetilde{\tau}^{\mathbf{U}^N} \wedge \widetilde{\tau}^{\mathbf{V}^N} \wedge t_{\varepsilon,M}]$. But it appears that the case $\widetilde{\tau}^{\mathbf{U}^N} \wedge \widetilde{\tau}^{\mathbf{V}^N} < t_{\varepsilon,M}$ is impossible for small values of ε. Indeed, it would either imply $\|(\mathbf{U}^N - \mathbf{u}^N)(\widetilde{\tau}^{\mathbf{U}^N})\|_{\ell^1_0(\mathbb{T}_N)} \le \varepsilon/2$ or $\|(\mathbf{V}^N - \mathbf{v}^N)(\widetilde{\tau}^{\mathbf{V}^N})\|_{\ell^1_0(\mathbb{T}_N)} \le \varepsilon/2$ which is impossible because we have on the one hand

$$
\left\|\mathbf{U}^N\left(\widetilde{\tau}^{\mathbf{U}^N}\right)\right\|_{\ell^1_0(\mathbb{T}_N)} \ge M + \varepsilon \qquad \left(\text{or } \left\|\mathbf{V}^N\left(\widetilde{\tau}^{\mathbf{V}^N}\right)\right\|_{\ell^1_0(\mathbb{T}_N)} \ge M + \varepsilon\right),
$$

and on the other hand

$$
\left\|\mathbf{u}^N\left(\widetilde{\tau}^{\mathbf{U}^N}\right)\right\|_{\ell^1_0(\mathbb{T}_N)} \le \left\|\mathbf{u}^N\left(\widetilde{\tau}^{\mathbf{U}^N}\right)\right\|_{\ell^2_0(\mathbb{T}_N)} \le \|\mathbf{u}_0\|_{\ell^2_0(\mathbb{T}_N)} \le M \qquad \left(\text{or } \left\|\mathbf{v}^N\left(\widetilde{\tau}^{\mathbf{V}^N}\right)\right\|_{\ell^1_0(\mathbb{T}_N)} \le M\right).
$$

Therefore, Inequality (56) holds for all $t \in [0, t_{\varepsilon,M}]$. Thus,

$$
\begin{aligned}
\left\|\mathbf{U}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} &\le \left\|\mathbf{U}^N(t_{\varepsilon,M}) - \mathbf{u}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(t_{\varepsilon,M}) - \mathbf{v}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} \\
&\quad + \left\|\mathbf{u}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{v}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} \\
&\le \varepsilon,
\end{aligned}
$$

and we have just shown that

$$
\left\{\sup_{t\in[0,t_{\varepsilon,M}]} \left\|\mathbf{W}^{Q,N}(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \le \delta_\varepsilon\right\} \subset \left\{\left\|\mathbf{U}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} \le \varepsilon\right\}.
$$

and therefore,

$$
\mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}\left(\left\|\mathbf{U}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(t_{\varepsilon,M})\right\|_{\ell^1_0(\mathbb{T}_N)} \le \varepsilon\right) \ge \mathbb{P}\left(\sup_{t\in[0,t_{\varepsilon,M}]} \left\|\mathbf{W}^{Q,N}(t)\right\|_{\ell^1_0(\mathbb{T}_N)} \le \delta_\varepsilon\right).
$$

Notice that the right-hand side does not depend on $\mathbf{u}_0$ nor $\mathbf{v}_0$. Furthermore, it is positive since $\mathbf{W}^{Q,N}$ is an $\mathbb{R}^N$-valued Wiener process. Hence, taking the infimum over $\mathbf{u}_0$ and $\mathbf{v}_0$ on the left-hand side yields the wanted result. □

Proof of Lemma 2.11. From Itô's formula, we have for all $t \ge 0$,

$$
\begin{aligned}
&\left\|\mathbf{U}^N(\tau_M \wedge t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} + \left\|\mathbf{V}^N(\tau_M \wedge t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} \\
&= \|\mathbf{u}_0\|^2_{\ell^2_0(\mathbb{T}_N)} + \|\mathbf{v}_0\|^2_{\ell^2_0(\mathbb{T}_N)} + \int_0^{\tau_M\wedge t} \left\langle \mathbf{b}(\mathbf{U}^N(s)), \mathbf{U}^N(s)\right\rangle_{\ell^2_0(\mathbb{T}_N)} \mathrm{d}s + \int_0^{\tau_M\wedge t} \left\langle \mathbf{b}(\mathbf{V}^N(s)), \mathbf{V}^N(s)\right\rangle_{\ell^2_0(\mathbb{T}_N)} \mathrm{d}s \\
&\quad + \int_0^{\tau_M\wedge t} \left\langle \mathbf{U}^N(s) + \mathbf{V}^N(s), \mathrm{d}\mathbf{W}^{Q,N}(s)\right\rangle_{\ell^2_0(\mathbb{T}_N)} + 2\sum_{k\ge1} \int_0^{\tau_M\wedge t} \left\|\mathbf{g}^k\right\|^2_{\ell^2_0(\mathbb{T}_N)} \mathrm{d}s.
\end{aligned} \tag{57}
$$

The fifth term of the right-hand side is a martingale. Indeed, by the Cauchy–Schwarz inequality, Inequality (24), and the bound (27), we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k\geq 1}\int_0^{\tau_M\wedge t}\left|\left\langle \mathbf{U}^N(s)+\mathbf{V}^N(s),\mathbf{g}^k\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right|^2\mathrm{d}s\right]\\
&\leq\left(\sum_{k\geq 1}\|\mathbf{g}^k\|^2_{\ell_0^2(\mathbb{T}_N)}\right)\mathbb{E}\left[\int_0^t\|\mathbf{U}^N(s)+\mathbf{V}^N(s)\|^2_{\ell_0^2(\mathbb{T}_N)}\mathrm{d}s\right]\\
&\leq 2\mathsf{D}\left(\mathbb{E}\left[\int_0^t\|\mathbf{U}^N(s)\|^2_{\ell_0^2(\mathbb{T}_N)}\mathrm{d}s\right]+\mathbb{E}\left[\int_0^t\|\mathbf{V}^N(s)\|^2_{\ell_0^2(\mathbb{T}_N)}\mathrm{d}s\right]\right)\\
&\leq 2\mathsf{D}\left(2\mathsf{c}_0^{(2)}+\mathsf{c}_1^{(2)}\left(\|\mathbf{u}_0\|^2_{\ell_0^2(\mathbb{T}_N)}+\|\mathbf{v}_0\|^2_{\ell_0^2(\mathbb{T}_N)}\right)+2\mathsf{c}_2^{(2)}t\right)\\
&<+\infty.
\end{aligned}$$

Thus, taking the expectation in (57), applying Lemma 2.1.(ii), Inequality (24), (22) and (29), we get

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbf{U}^N(\tau_M\wedge t)\|^2_{\ell_0^2(\mathbb{T}_N)}+\|\mathbf{V}^N(\tau_M\wedge t)\|^2_{\ell_0^2(\mathbb{T}_N)}\right]-\left(\|\mathbf{u}_0\|^2_{\ell_0^2(\mathbb{T}_N)}+\|\mathbf{v}_0\|^2_{\ell_0^2(\mathbb{T}_N)}\right)\\
&=2\mathbb{E}\left[\int_0^{\tau_M\wedge t}\left(\left\langle\mathbf{b}\left(\mathbf{U}^N(s)\right),\mathbf{U}^N(s)\right\rangle_{\ell_0^2(\mathbb{T}_N)}+\left\langle\mathbf{b}\left(\mathbf{V}^N(s)\right),\mathbf{V}^N(s)\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right)\mathrm{d}s\right]+2\mathbb{E}\left[\int_0^{\tau_M\wedge t}\sum_{k\geq 1}\|\mathbf{g}^k\|^2_{\ell_0^2(\mathbb{T}_N)}\mathrm{d}s\right]\\
&\leq -2\nu\mathbb{E}\left[\int_0^{\tau_M\wedge t}\left(\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}^N(s)\right\|^2_{\ell_0^2(\mathbb{T}_N)}+\left\|\mathbf{D}_N^{(1,+)}\mathbf{V}^N(s)\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right)\mathrm{d}s\right]+2\mathbb{E}\left[\tau_M\wedge t\right]\mathsf{D}\\
&\leq -2\nu\mathbb{E}\left[\int_0^{\tau_M\wedge t}\left(\|\mathbf{U}^N(s)\|^2_{\ell_0^2(\mathbb{T}_N)}+\|\mathbf{V}^N(s)\|^2_{\ell_0^2(\mathbb{T}_N)}\right)\mathrm{d}s\right]+2\mathbb{E}\left[\tau_M\wedge t\right]\mathsf{D}\\
&\leq 2\left(\mathsf{D}-\nu M^2\right)\mathbb{E}\left[\tau_M\wedge t\right].
\end{aligned}$$

So if we choose $M>\sqrt{\mathsf{D}/\nu}$, we get

$$\mathbb{E}[\tau_M\wedge t]\leq\frac{\|\mathbf{u}_0\|^2_{\ell_0^2(\mathbb{T}_N)}+\|\mathbf{v}_0\|^2_{\ell_0^2(\mathbb{T}_N)}}{2\left(\nu M^2-\mathsf{D}\right)},$$

and we deduce from the monotone convergence theorem that $\mathbb{E}[\tau_M]=\lim_{t\to\infty}\mathbb{E}[\tau_M\wedge t]<+\infty$. □

B.2. **The split-step scheme.**

Proof of Lemma 2.18. First, let $\varepsilon>0$ and let us fix $\mathbf{u}_0,\mathbf{v}_0\in\mathbb{R}_0^N$ such that $\|\mathbf{u}_0\|_{\ell_0^2(\mathbb{T}_N)}\leq M$ and $\|\mathbf{v}_0\|_{\ell_0^2(\mathbb{T}_N)}\leq M$.

Let $(\mathbf{u}_n^{N,\Delta t})_{n\in\mathbb{N}}$ and $(\mathbf{v}_n^{N,\Delta t})_{n\in\mathbb{N}}$ denote the noiseless counterparts of the sequences $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ and $(\mathbf{V}_n^{N,\Delta t})_{n\in\mathbb{N}}$, *i.e.*

$$\mathbf{u}_{n+1}^{N,\Delta t}=\mathbf{u}_n^{N,\Delta t}+\Delta t\mathbf{b}\left(\mathbf{u}_{n+1}^{N,\Delta t}\right),\qquad\mathbf{v}_{n+1}^{N,\Delta t}=\mathbf{v}_n^{N,\Delta t}+\Delta t\mathbf{b}\left(\mathbf{v}_{n+1}^{N,\Delta t}\right),\tag{58}$$

with initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$. Then $(\mathbf{u}_n^{N,\Delta t})_{n\in\mathbb{N}}$ and $(\mathbf{v}_n^{N,\Delta t})_{n\in\mathbb{N}}$ are subject to non-perturbed $\ell_0^2(\mathbb{T}_N)$ dissipativity, and consequently the sum of their energies decreases to 0 over time. Indeed, we have

$$\begin{aligned}
\|\mathbf{u}_n^{N,\Delta t}\|^2_{\ell_0^2(\mathbb{T}_N)}+\|\mathbf{v}_n^{N,\Delta t}\|^2_{\ell_0^2(\mathbb{T}_N)}&=\left\|\mathbf{u}_{n+1}^{N,\Delta t}-\Delta t\mathbf{b}\left(\mathbf{u}_{n+1}^{N,\Delta t}\right)\right\|^2_{\ell_0^2(\mathbb{T}_N)}+\left\|\mathbf{v}_{n+1}^{N,\Delta t}-\Delta t\mathbf{b}\left(\mathbf{v}_{n+1}^{N,\Delta t}\right)\right\|^2_{\ell_0^2(\mathbb{T}_N)}\\
&=\left\|\mathbf{u}_{n+1}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}+\left\|\mathbf{v}_{n+1}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}+(\Delta t)^2\left(\|\mathbf{b}\left(\mathbf{u}_{n+1}\right)\|^2_{\ell_0^2(\mathbb{T}_N)}+\|\mathbf{b}\left(\mathbf{v}_{n+1}\right)\|^2_{\ell_0^2(\mathbb{T}_N)}\right)\\
&\quad-2\Delta t\left(\left\langle\mathbf{u}_{n+1}^{N,\Delta t},\mathbf{b}\left(\mathbf{u}_{n+1}^{N,\Delta t}\right)\right\rangle_{\ell_0^2(\mathbb{T}_N)}+\left\langle\mathbf{v}_{n+1}^{N,\Delta t},\mathbf{b}\left(\mathbf{v}_{n+1}^{N,\Delta t}\right)\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right)
\end{aligned}$$

therefore, using successively Lemma 2.1.(ii) and (22), we get

$$\begin{aligned}
&\left\|\mathbf{u}_{n+1}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}+\left\|\mathbf{v}_{n+1}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}-\left(\|\mathbf{u}_n^{N,\Delta t}\|^2_{\ell_0^2(\mathbb{T}_N)}+\|\mathbf{v}_n^{N,\Delta t}\|^2_{\ell_0^2(\mathbb{T}_N)}\right)\\
&\leq 2\Delta t\left(\left\langle\mathbf{u}_{n+1}^{N,\Delta t},\mathbf{b}\left(\mathbf{u}_{n+1}^{N,\Delta t}\right)\right\rangle_{\ell_0^2(\mathbb{T}_N)}+\left\langle\mathbf{v}_{n+1}^{N,\Delta t},\mathbf{b}\left(\mathbf{v}_{n+1}^{N,\Delta t}\right)\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right)\\
&\leq -2\Delta t\nu\left(\left\|\mathbf{D}_N^{(1,+)}\mathbf{u}_{n+1}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}+\left\|\mathbf{D}_N^{(1,+)}\mathbf{v}_{n+1}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right)\\
&\leq -2\Delta t\nu\left(\left\|\mathbf{u}_{n+1}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}+\left\|\mathbf{v}_{n+1}^{N,\Delta t}\right\|^2_{\ell_0^2(\mathbb{T}_N)}\right)
\end{aligned}$$

so that

$$\left\|\mathbf{u}_{n+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2+\left\|\mathbf{v}_{n+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \frac{1}{1+2\Delta t\nu}\left(\left\|\mathbf{u}_{n}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2+\left\|\mathbf{v}_{n}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right),$$

by induction, we get for all $n\in\mathbb{N}$,

$$\left\|\mathbf{u}_{n}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2+\left\|\mathbf{v}_{n}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \left(\frac{1}{1+2\Delta t\nu}\right)^n\left(\left\|\mathbf{u}_0\right\|_{\ell_0^2(\mathbb{T}_N)}^2+\left\|\mathbf{v}_0\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right).$$

It appears now that if we fix the value

$$n_{\varepsilon,M}:=\left\lceil\frac{-1}{\log(1+2\Delta t\nu)}\log\left(\frac{\varepsilon^2}{16M^2}\right)\right\rceil,$$

we get for all $n\geq n_{\varepsilon,M}$,

$$\left\|\mathbf{u}_{n}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{v}_{n}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}\leq\left\|\mathbf{u}_{n}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}+\left\|\mathbf{v}_{n}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}\leq\frac{\varepsilon}{2}.$$

Now, we fix $\delta_\varepsilon:=\varepsilon/(4n_{\varepsilon,M})$ and we restrict ourselves to the event

$$\left\{\sup_{n=1,\dots,n_{\varepsilon,M}}\left\|\Delta\mathbf{W}_n^{Q,N}\right\|_{\ell_0^1(\mathbb{T}_N)}\leq\delta_\varepsilon\right\}. \tag{59}$$

Let $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ and $(\mathbf{V}_n^{N,\Delta t})_{n\in\mathbb{N}}$ be two solutions of (14) with the deterministic initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$ respectively. With similar arguments as for the proof of Proposition 2.7, we get from (14), (58) and Lemma 2.1.(ii), for all $n\in\mathbb{N}$,

$$\begin{aligned}
&\left\|\mathbf{U}_{n+1}^{N,\Delta t}-\mathbf{u}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n+1}^{N,\Delta t}-\mathbf{v}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}\\
&\leq\left\|\mathbf{U}_{n+\frac12}^{N,\Delta t}-\mathbf{u}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n+\frac12}^{N,\Delta t}-\mathbf{v}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+2\left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_{\ell_0^1(\mathbb{T}_N)}\\
&=\left\langle\mathbf{sign}\left(\mathbf{U}_{n+\frac12}^{N,\Delta t}-\mathbf{u}_{n+1}^{N,\Delta t}\right),\mathbf{U}_{n}^{N,\Delta t}-\mathbf{u}_{n}^{N,\Delta t}\right\rangle_{\ell^2(\mathbb{T}_N)}\\
&\quad+\Delta t\left\langle\mathbf{sign}\left(\mathbf{U}_{n+\frac12}^{N,\Delta t}-\mathbf{u}_{n+1}^{N,\Delta t}\right),\mathbf{b}\left(\mathbf{U}_{n+\frac12}^{N,\Delta t}\right)-\mathbf{b}\left(\mathbf{u}_{n+1}^{N,\Delta t}\right)\right\rangle_{\ell^2(\mathbb{T}_N)}\\
&\quad+\left\langle\mathbf{sign}\left(\mathbf{V}_{n+\frac12}^{N,\Delta t}-\mathbf{v}_{n+1}^{N,\Delta t}\right),\mathbf{V}_{n}^{N,\Delta t}-\mathbf{v}_{n}^{N,\Delta t}\right\rangle_{\ell^2(\mathbb{T}_N)}\\
&\quad+\Delta t\left\langle\mathbf{sign}\left(\mathbf{V}_{n+\frac12}^{N,\Delta t}-\mathbf{v}_{n+1}^{N,\Delta t}\right),\mathbf{b}\left(\mathbf{V}_{n+\frac12}^{N,\Delta t}\right)-\mathbf{b}\left(\mathbf{v}_{n+1}^{N,\Delta t}\right)\right\rangle_{\ell^2(\mathbb{T}_N)}+2\left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_{\ell_0^1(\mathbb{T}_N)}\\
&\leq\left\|\mathbf{U}_{n}^{N,\Delta t}-\mathbf{u}_{n}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n}^{N,\Delta t}-\mathbf{v}_{n}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+2\left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_{\ell_0^1(\mathbb{T}_N)}.
\end{aligned}$$

On the event (59), we have for all $n=1,\dots,n_{\varepsilon,M}$,

$$\left\|\mathbf{U}_{n+1}^{N,\Delta t}-\mathbf{u}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n+1}^{N,\Delta t}-\mathbf{v}_{n+1}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}\leq\left\|\mathbf{U}_{n}^{N,\Delta t}-\mathbf{u}_{n}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n}^{N,\Delta t}-\mathbf{v}_{n}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+2\delta_\varepsilon.$$

In particular, by induction, we have

$$\left\|\mathbf{U}_{n_{\varepsilon,M}}^{N,\Delta t}-\mathbf{u}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n_{\varepsilon,M}}^{N,\Delta t}-\mathbf{v}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}\leq 2n_{\varepsilon,M}\delta_\varepsilon=\frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned}
\left\|\mathbf{U}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}&\leq\left\|\mathbf{U}_{n_{\varepsilon,M}}^{N,\Delta t}-\mathbf{u}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n_{\varepsilon,M}}^{N,\Delta t}-\mathbf{v}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{u}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{v}_{n_{\varepsilon,M}}^{N,\Delta t}\right\|_{\ell_0^1(\mathbb{T}_N)}\\
&\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

We just have shown that

$$\mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}\left(\left\|\mathbf{U}_{n_{\varepsilon,M}}\right\|_{\ell_0^1(\mathbb{T}_N)}+\left\|\mathbf{V}_{n_{\varepsilon,M}}\right\|_{\ell_0^1(\mathbb{T}_N)}\leq\varepsilon\right)\geq\mathbb{P}\left(\sup_{n=1,\dots,n_{\varepsilon,M}}\left\|\Delta\mathbf{W}_n^{Q,N}\right\|_{\ell_0^1(\mathbb{T}_N)}\leq\delta_\varepsilon\right)>0.$$

Since the event (59) does not depend on $\mathbf{u}_0$ nor $\mathbf{v}_0$, we get the result. ☐

Appendix C. Proofs of finite-time bounds for Sections 3 and 4

Proof of Lemma 3.6. The proof of these estimates is largely based on refinements of computations made in [28].

Proof of the $L^p_0(\mathbb{T})$ estimate. Let $p \in 2\mathbb{N}^*$ and let us repeat the proof of [28, Lemma 3] up to [28, Equation (23)]. When the initial condition u^*_0 is random and has distribution μ^*, this equation writes

$$\mathbb{E}\left[\|u^*(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right] = \mathbb{E}\left[\|u^*_0\|^p_{L^p_0(\mathbb{T})}\right] - p\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} \partial_x A(u^*(s))u^*(s)^{p-1}\mathrm{d}x\mathrm{d}s\right]$$
$$- \nu p(p-1)\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} \partial_x u^*(s)^2 u^*(s)^{p-2}\mathrm{d}x\mathrm{d}s\right] + \frac{p(p-1)}{2}\sum_{k\geq 1}\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} u^*(s)^{p-2}(g^k)^2\mathrm{d}x\mathrm{d}s\right],$$

for all $t \geq 0$ and $r \geq 0$, where T_r is a stopping time converging almost surely towards $+\infty$ as $r \to +\infty$ (by [28, Corollary 2]). Using [28, Equation (24)], the non-positivity of the third term of the right-hand side, and bounding the g^k's by their supremum, we get the inequality

$$\mathbb{E}\left[\|u^*(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right] \leq \mathbb{E}\left[\|u^*_0\|^p_{L^p_0(\mathbb{T})}\right] + \frac{p(p-1)}{2}\left(\sum_{k\geq 1}\left\|g^k\right\|^2_{L^\infty_0(\mathbb{T})}\right)\mathbb{E}\left[\int_0^{t\wedge T_r}\|u^*(s)\|^{p-2}_{L^{p-2}_0(\mathbb{T})}\,\mathrm{d}s\right].$$

Using now Corollary 3.3, (3), (5), and [28, Equation (18)], we get

$$\mathbb{E}\left[\|u^*(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right] \leq \mathsf{C}^{0,p} + \frac{p(p-1)}{2}\mathsf{D}\left(C_5^{(p-2)}\left(1+\mathbb{E}\left[\|u^*_0\|^{p-2}_{L^{p-2}_0(\mathbb{T})}\right]\right) + C_6^{(p-2)}t\right),$$

where the constants $C_5^{(p-2)}$ and $C_6^{(p-2)}$, defined in [28], depend only on ν, p and D. Using once again Corollary 3.3 and letting $r \to +\infty$, we obtain

$$\limsup_{r\to\infty}\mathbb{E}\left[\|u^*(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right] \leq C^{0,p} + \frac{p(p-1)}{2}\mathsf{D}\left(C_5^{(p-2)}\left(1+\mathsf{C}^{0,p-2}\right) + C_6^{(p-2)}t\right) =: \mathsf{C}^{*,0,p}_t.$$

Applying Fatou's lemma on the left-hand side, we get

$$\mathbb{E}\left[\|u^*(t)\|^p_{L^p_0(\mathbb{T})}\right] \leq \mathsf{C}^{*,0,p}_t,$$

from which we easily get the claimed inequality in the case $p \in 2\mathbb{N}^*$. The general case $p \in [1,+\infty)$ then follows from the Jensen inequality.

Proof of the $H^1_0(\mathbb{T})$ and $H^2_0(\mathbb{T})$ estimates. We now start from [28, Lemma 4] which, when u^*_0 is random, gives the estimate

$$\mathbb{E}\left[\|u^*(t\wedge T_r)\|^2_{H^1_0(\mathbb{T})}\right] + \nu\mathbb{E}\left[\int_0^{t\wedge T_r}\|u^*(s)\|^2_{H^2_0(\mathbb{T})}\,\mathrm{d}s\right] \leq \mathbb{E}\left[\|u^*_0\|^2_{H^1_0(\mathbb{T})}\right] + C_7\left(1+\mathbb{E}\left[\|u^*_0\|^{2\mathsf{p}_A+2}_{L^{2\mathsf{p}_A+2}_0(\mathbb{T})}\right]\right) + C_8 t,$$

and from which we deduce, by applying Fatou's lemma on the left-hand side and Corollary 3.3 on the right-hand side:

$$\mathbb{E}\left[\|u^*(t)\|^2_{H^1_0(\mathbb{T})}\right] + \nu\mathbb{E}\left[\int_0^t\|u^*(s)\|^2_{H^2_0(\mathbb{T})}\,\mathrm{d}s\right] \leq \mathsf{C}^{1,2} + C_7\left(1+\mathsf{C}^{0,2\mathsf{p}_A+2}\right) + C_8 t =: \mathsf{C}^{*,1,2}_t.$$

We conclude that

$$\mathbb{E}\left[\|u^*(t)\|^2_{H^1_0(\mathbb{T})}\right] \leq \mathsf{C}^{1,2} + C_7\left(1+\mathsf{C}^{0,2\mathsf{p}_A+2}\right) + C_8 t =: \mathsf{C}^{*,1,2}_t,$$

and

$$\mathbb{E}\left[\int_0^t\|u^*(s)\|^2_{H^2_0(\mathbb{T})}\,\mathrm{d}s\right] \leq \frac{1}{\nu}\left(\mathsf{C}^{1,2} + C_7\left(1+\mathsf{C}^{0,2\mathsf{p}_A+2}\right) + C_8 t\right) =: \mathsf{C}^{*,2,2}_t. \qquad \square$$

Proof of Lemma 3.9. Lemma 3.9 is a refinement of the uniform $h^2_0(\mathbb{T}_N)$ estimate from Proposition 3.1. We start from (39) and use the definition of $\mathbf{b}$ to write

$$\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}^N(t)\right\|^2_{\ell^2_0(\mathbb{T}_N)} = \left\|\mathbf{D}_N^{(1,+)}\mathbf{U}^N_0\right\|^2_{\ell^2_0(\mathbb{T}_N)} + 2\int_0^t\left\langle\mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathbf{D}_N^{(1,+)}\left(-\mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(s)) + \nu\mathbf{D}_N^{(2)}\mathbf{U}^N(s)\right)\right\rangle_{\ell^2_0(\mathbb{T}_N)}\mathrm{d}s$$
$$+ 2\int_0^t\left\langle\mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathrm{d}\left(\mathbf{D}_N^{(1,+)}\mathbf{W}^{Q,N}\right)(s)\right\rangle_{\ell^2_0(\mathbb{T}_N)} + t\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1,+)}\mathbf{g}^k\right\|^2_{\ell^2_0(\mathbb{T}_N)}.$$

By (20) and Young's inequality,

$$-2\left\langle\mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathbf{D}_N^{(1,+)}\mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(s))\right\rangle_{\ell^2_0(\mathbb{T}_N)} = 2\left\langle\mathbf{D}_N^{(2)}\mathbf{U}^N(s), \mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(s))\right\rangle_{\ell^2_0(\mathbb{T}_N)}$$
$$\leq 2\nu\|\mathbf{D}_N^{(2)}\mathbf{U}^N(s)\|^2_{\ell^2_0(\mathbb{T}_N)} + \frac{1}{2\nu}\|\mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(s))\|^2_{\ell^2_0(\mathbb{T}_N)},$$

while

$$2\nu\left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathbf{D}_N^{(1,+)}\mathbf{D}_N^{(2)}\mathbf{U}^N(s)\right\rangle_{\ell_0^2(\mathbb{T}_N)} = -2\nu\|\mathbf{D}_N^{(2)}\mathbf{U}^N(s)\|_{\ell_0^2(\mathbb{T}_N)}^2,$$

so that, using (24) in addition,

$$\begin{aligned}\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}^N(t)\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_0^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + \frac{1}{2\nu}\int_0^t \|\mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(s))\|_{\ell_0^2(\mathbb{T}_N)}^2 \mathrm{d}s \\ + 2\int_0^t \left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathrm{d}\left(\mathbf{D}_N^{(1,+)}\mathbf{W}^{Q,N}\right)(s)\right\rangle_{\ell_0^2(\mathbb{T}_N)} + t\mathsf{D}.\end{aligned}$$

We deduce that for any $t \geq 0$,

$$\sup_{s\in[0,t]}\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}^N(s)\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \leq \left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_0^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + \frac{1}{2\nu}\int_0^t \|\mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(s))\|_{\ell_0^2(\mathbb{T}_N)}^2 \mathrm{d}s + 2M_t^N + t\mathsf{D}, \tag{60}$$

where M_t^N is defined by

$$M_t^N = \sup_{s\in[0,t]}\int_0^s \left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(r), \mathrm{d}\left(\mathbf{D}_N^{(1,+)}\mathbf{W}^{Q,N}\right)(r)\right\rangle_{\ell_0^2(\mathbb{T}_N)},$$

and it remains to control the expectation of the right-hand side of the inequality (60).

First, by Proposition 3.1, we have

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_0^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq \mathsf{C}^{1,2}.$$

Next, by stationarity of $\mathbf{U}^N$ and (41), we have

$$\mathbb{E}\left[\int_0^t \|\mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}^N(s))\|_{\ell_0^2(\mathbb{T}_N)}^2 \mathrm{d}s\right] \leq t\mathbb{E}\left[\|\mathbf{D}^{(1,-)}\overline{\mathbf{A}}^N(\mathbf{U}_0^N)\|_{\ell_0^2(\mathbb{T}_N)}^2\right] \leq t8\mathsf{C}_{\bar{A}}^2\left(\mathsf{C}^{1,2} + \frac{\mathsf{D}}{2\nu}\mathsf{C}^{0,2\mathsf{p}_{\bar{A}}}\right).$$

Finally, we recall that by (40), the process $(\int_0^t \langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(s), \mathrm{d}(\mathbf{D}_N^{(1,+)}\mathbf{W}^{Q,N})(s)\rangle_{\ell_0^2(\mathbb{T}_N)})_{t\geq 0}$ is a martingale. Therefore, applying successively the Jensen and the Doob inequalities, the Itô isometry, the Cauchy–Schwarz inequality, Proposition 3.1 and (24), we get

$$\begin{aligned}\mathbb{E}\left[M_t^N\right] &\leq \mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s \left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(r), \mathrm{d}\left(\mathbf{D}_N^{(1,+)}\mathbf{W}^{Q,N}\right)(r)\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right|^2\right]^{1/2} \\ &\leq 2\mathbb{E}\left[\left|\int_0^t \left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(r), \mathrm{d}\left(\mathbf{D}_N^{(1,+)}\mathbf{W}^{Q,N}\right)(r)\right\rangle_{\ell_0^2(\mathbb{T}_N)}\right|^2\right]^{1/2} \\ &= 2\mathbb{E}\left[\sum_{k\geq 1}\int_0^t \left\langle \mathbf{D}_N^{(1,+)}\mathbf{U}^N(r), \mathbf{D}_N^{(1,+)}\mathbf{g}^k\right\rangle_{\ell_0^2(\mathbb{T}_N)}^2 \mathrm{d}r\right]^{1/2} \\ &\leq 2\sqrt{t}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_0^N\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right]^{1/2}\left(\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1,+)}\mathbf{g}^k\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right)^{1/2} \\ &\leq 2\sqrt{t\mathsf{C}^{1,2}\mathsf{D}},\end{aligned}$$

which completes the proof. □

Proof of Lemma 4.6. Lemma 4.6 is a refinement of the proof of Proposition 2.17. We fix $t > 0$, and for the sake of simplicity we write Δt in place of Δt_j. We also introduce the notation $n_t = \lfloor \frac{t}{\Delta t}\rfloor$.

We start from Equation (34). For all $n = 0, \ldots, n_t$, we write

$$\begin{aligned}\|\mathbf{U}_n^{N,\Delta t}\|_{\ell_0^2(\mathbb{T}_N)}^2 &= \left\|\mathbf{U}_0^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + \sum_{l=0}^{n-1}\left(\left\|\mathbf{U}_{l+1}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 - \left\|\mathbf{U}_l^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2\right) \\ &\leq \left\|\mathbf{U}_0^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 - 2\nu\Delta t\sum_{l=0}^{n-1}\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + 2\sum_{l=0}^{n-1}\left\langle \mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle_{\ell_0^2(\mathbb{T}_N)} + \sum_{l=0}^{n-1}\left\|\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 \\ &\leq \left\|\mathbf{U}_0^{N,\Delta t}\right\|_{\ell_0^2(\mathbb{T}_N)}^2 + 2\sum_{l=0}^{n-1}\left\langle \mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle_{\ell_0^2(\mathbb{T}_N)} + \sum_{l=0}^{n-1}\left\|\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_{\ell_0^2(\mathbb{T}_N)}^2.\end{aligned}$$

Taking the supremum in time and the expectation, we get

$$\mathbb{E}\left[\sup_{n=0,\dots,n_t}\left\|\mathbf{U}_n^{N,\Delta t}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\leq \mathbb{E}\left[\left\|\mathbf{U}_0^{N,\Delta t}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]+2\mathbb{E}\left[\sup_{n=0,\dots,n_t}\left|\sum_{l=0}^{n-1}\left\langle\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t},\Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle_{\ell^2_0(\mathbb{T}_N)}\right|\right]+\mathbb{E}\left[\sum_{l=0}^{n_t-1}\left\|\Delta\mathbf{W}_{l+1}^{Q,N}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]. \tag{61}$$

First, by (22) and Proposition 4.1, we have

$$\mathbb{E}\left[\left\|\mathbf{U}_0^{N,\Delta t}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\leq \mathbb{E}\left[\left\|\mathbf{D}_N^{(1,+)}\mathbf{U}_0^{N,\Delta t}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\leq \mathsf{C}^{\Delta,1,2}.$$

Noticing that the sequence $(\sum_{l=0}^{n-1}\langle\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t},\Delta\mathbf{W}_{l+1}^{Q,N}\rangle_{\ell^2_0(\mathbb{T}_N)})_{n\geq 1}$ is a martingale, we get by applying successively Jensen's and Doob's inequalities to the second term of the right-hand side,

$$\begin{aligned}\mathbb{E}\left[\sup_{n=0,\dots,n_t}\left|\sum_{l=0}^{n-1}\left\langle\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t},\Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle_{\ell^2_0(\mathbb{T}_N)}\right|\right] &\leq \mathbb{E}\left[\sup_{n=0,\dots,n_t}\left|\sum_{l=0}^{n-1}\left\langle\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t},\Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle_{\ell^2_0(\mathbb{T}_N)}\right|^2\right]^{1/2}\\ &\leq 2\mathbb{E}\left[\left|\sum_{l=0}^{n_t-1}\left\langle\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t},\Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle_{\ell^2_0(\mathbb{T}_N)}\right|^2\right]^{1/2}.\end{aligned}$$

From (14), we may observe that each increment $\Delta\mathbf{W}_{l+1}^{Q,N}$ is independent from the family $(\mathbf{U}_{m+\frac{1}{2}}^{N,\Delta t},\Delta\mathbf{W}_m^{Q,N})_{m=0,\dots,l}$. Therefore, defining $\mathbf{V}^{N,\Delta t}$ and $\mathbf{V}_{\frac{1}{2}}^{N,\Delta t}$ as in Proposition 4.1, we have

$$\begin{aligned}2\mathbb{E}\left[\sum_{l=0}^{n_t-1}\left|\left\langle\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t},\Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle_{\ell^2_0(\mathbb{T}_N)}\right|^2\right]^{1/2} &\leq 2\left(\sum_{l=0}^{n_t-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\mathbb{E}\left[\left\|\Delta\mathbf{W}_{l+1}^{Q,N}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\right)^{1/2}\\ &\leq 2\sqrt{\mathsf{D}\Delta t}\left(n_t\mathbb{E}\left[\left\|\mathbf{V}_{\frac{1}{2}}^{N,\Delta t}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\right)^{1/2}\\ &\leq 2\sqrt{\mathsf{D}t\mathsf{C}_{\frac{1}{2}}^{\Delta,1,2}},\end{aligned}$$

where we have used (35) at the second line and Proposition 4.1 together with (22) at the third line. Injecting this bound into (61), and using (35) again, we finally get

$$\mathbb{E}\left[\sup_{n=0,\dots,n_t}\left\|\mathbf{U}_n^{N,\Delta t}\right\|^2_{\ell^2_0(\mathbb{T}_N)}\right]\leq \mathsf{C}^{\Delta,1,2}+2\sqrt{\mathsf{D}t\mathsf{C}_{\frac{1}{2}}^{\Delta,1,2}}+t\mathsf{D}=:\mathsf{S}_t^{\Delta,0,2}.$$

□

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Sébastien Boyaval
LHSV, Ecole des Ponts, CEREMA, EDF R et D, Chatou, France
Inria Paris, Matherials
Email address: sebastien.boyaval@enpc.fr

Sofiane Martel
Inria Rennes - Bretagne Atlantique, SIMSMART
Email address: sofiane.martel@inria.fr

Julien Reygner
CERMICS, Ecole des Ponts, Marne-la-Vallée, France
Email address: julien.reygner@enpc.fr